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MAPPINGS PRESERVING SUM OF PRODUCTS
 $ab - b \circ a^*$ ON FACTOR VON NEUMANN ALGEBRAS

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Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. In this paper, we proved that a bijective mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ satisfies $\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*$ (where $\circ$ is the special Jordan product on $\mathcal{A}$ and $\mathcal{B}$, respectively), for all elements $a, b \in \mathcal{A}$, if and only if Φ is a $*$ -ring isomorphism. In particular, if the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are type I factors, then Φ is a unitary isomorphism or a conjugate unitary isomorphism.

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1 Introduction

Let $\mathcal{R}$ and $\mathcal{S}$ be rings. We say that a mapping $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ *preserves product* if $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathcal{R}$, *preserves Jordan product* if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$, for all elements $a, b \in \mathcal{R}$, and *preserves Lie product* if $\Phi(ab - ba) = \Phi(a)\Phi(b) - \Phi(b)\Phi(a)$, for all elements $a, b \in \mathcal{R}$. We say that a mapping $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ is *additive* if $\Phi(a + b) = \Phi(a) + \Phi(b)$, for all elements $a, b \in \mathcal{R}$ and that is a *ring isomorphism* if Φ is an additive bijection that preserves products.

Let $\mathcal{R}$ and $\mathcal{S}$ be $*$ -rings. We say that a mapping $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ *preserves involution* if $\Phi(a^*) = \Phi(a)^*$, for all elements $a \in \mathcal{R}$, and that Φ is a *$*$ -ring isomorphism* if Φ is a ring isomorphism that preserves involution.

In recent years, there has been a great interest in the study of additivity of mappings. A part of these results is focused on the additivity of mappings that preserve products, Jordan products and Lie products on rings as well as operator algebras (for example, see [2], [5], [6] and [7]).

As in the case of rings, a natural problem on $*$ -rings is to know when mappings defined on them preserving some new type of product are $*$ -ring isomorphisms. Recently, many mathematicians devoted themselves to study mappings preserving new products on rings with involution (for example, see [1], [3] and [4]). In particular, Cui and Li [1] studied the bijective mappings preserving the new product $ab - ba^*$. They showed that such mappings on factor von Neumann algebras are $*$ -ring isomorphisms.

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively, and $\mathcal{A}$ and $\mathcal{B}$ be two algebras over $\mathbb{C}$. If $a, b \in \mathcal{A}$, let $a \circ b = \frac{1}{2}(ab + ba)$. We call $\circ$ the *special Jordan product* on $\mathcal{A}$. We say that a mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ *preserves special Jordan product* if $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$, for all elements $a, b \in \mathcal{A}$.

A *von Neumann algebra* $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$, which are denote in this paper by lowercase Latin letters, containing the identity operator $1_{\mathcal{A}}$, called *elements of $\mathcal{A}$* . A von Neumann algebra $\mathcal{A}$ is called *factor* if its center contains the multiples of the identity operator only. It is well known that the factor von Neumann algebra $\mathcal{A}$ is prime, that is, if a and b are elements of $\mathcal{A}$ such that $a\mathcal{A}b = 0$, then either $a = 0$ or $b = 0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$ -algebras. We say that a mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ preserves sum of products $ab - b \circ a^*$ if

$$\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*, \quad (1)$$

for all elements $a, b \in \mathcal{A}$.

Similarly to the research performed in [1], the aim of the present paper is to investigate when a bijective mapping preserving sum of products $ab - b \circ a^*$ on factor von Neumann algebras is a $*$ -ring isomorphism. Our main result reads as follows.

Main Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ the identities of them, respectively. Then every bijective mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ satisfies $\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*$, for all elements $a, b \in \mathcal{A}$, if and only if Φ is a $*$ -ring isomorphism.*

2 The proof of the main theorem

The proof of the Main Theorem is made by proving several lemmas. We begin with the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras and $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ a mapping that preserve sum of products $ab - b \circ a^*$. If $\Phi(c) = \Phi(a) + \Phi(b)$, for elements a, b, c of $\mathcal{A}$, then hold the following identities: (i) $\Phi(tc - c \circ t^*) = \Phi(ta - a \circ t^*) + \Phi(tb - b \circ t^*)$ and (ii) $\Phi(ct - t \circ c^*) = \Phi(at - t \circ a^*) + \Phi(bt - t \circ b^*)$, for all elements t of $\mathcal{A}$.*

Proof. For any element t of $\mathcal{A}$, we have

$$\begin{aligned} \Phi(tc - c \circ t^*) &= \Phi(t)\Phi(c) - \Phi(c) \circ \Phi(t)^* \\ &= \Phi(t)(\Phi(a) + \Phi(b)) - (\Phi(a) + \Phi(b)) \circ \Phi(t)^* \\ &= \Phi(t)\Phi(a) - \Phi(a) \circ \Phi(t)^* + \Phi(t)\Phi(b) - \Phi(b) \circ \Phi(t)^* \\ &= \Phi(ta - a \circ t^*) + \Phi(tb - b \circ t^*). \end{aligned}$$

Correspondingly, we prove (ii). □

Lemma 2.2. $\Phi(0) = 0$.

Proof. From supposition of Φ , there is an element $b \in \mathcal{A}$ such that $\Phi(b) = 0$. This results that

$$\Phi(0) = \Phi(0b - b \circ 0^*) = \Phi(0)\Phi(b) - \Phi(b) \circ \Phi(0)^* = \Phi(0)0 - 0 \circ \Phi(0)^* = 0.$$

□

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $1_{\mathcal{A}}$ the identity of $\mathcal{A}$. Then every bijective mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ that preserve sum of products $ab - b \circ a^*$ is additive.*

We will establish the proof of Lemma 2.3 in a series of claims based on the techniques used by Cui and Li [1]. We begin, though, with a well-known result that will be used throughout this paper.

Let p_1 be an arbitrary non-trivial projection of $\mathcal{A}$ and write $p_2 = 1 - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ ($i, j = 1, 2$), satisfying the following multiplicative relations: $\mathcal{A}_{ij} \mathcal{A}_{kl} \subseteq \delta_{jk} \mathcal{A}_{il}$, where δ_{jk} is the Kronecker delta function.

Claim 2.1. *For any elements $a_{11} \in \mathcal{A}_{11}$, $a_{22} \in \mathcal{A}_{22}$, $b_{12} \in \mathcal{A}_{12}$ and $b_{21} \in \mathcal{A}_{21}$ hold: (i) $\Phi(a_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$, (ii) $\Phi(a_{11} + b_{21}) = \Phi(a_{11}) + \Phi(b_{21})$, (iii) $\Phi(a_{22} + b_{12}) = \Phi(a_{22}) + \Phi(b_{12})$ and (iv) $\Phi(a_{22} + b_{21}) = \Phi(a_{22}) + \Phi(b_{21})$.*

Proof. From supposition of Φ , consider an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ verifying $\Phi(c) = \Phi(a_{11}) + \Phi(b_{12})$. By Lemma 2.1(i) we have

$$\begin{aligned} \Phi(p_2 c - c \circ p_2^*) &= \Phi(p_2 a_{11} - a_{11} \circ p_2^*) + \Phi(p_2 b_{12} - b_{12} \circ p_2^*) \\ &= \Phi(p_2 b_{12} - b_{12} \circ p_2^*). \end{aligned}$$

On account of this we have that $p_2 c - c \circ p_2^* = -\frac{1}{2}b_{12}$ which implies that $c_{12} = b_{12}$ and $c_{21} = 0$. This shows that $\Phi(c_{11} + b_{12} + c_{22}) = \Phi(a_{11}) + \Phi(b_{12})$. Hence, for any element $d_{12} \in \mathcal{A}_{12}$ we have

$$\begin{aligned} \Phi(d_{12}(c_{11} + b_{12} + c_{22}) - (c_{11} + b_{12} + c_{22}) \circ d_{12}^*) &= \Phi(d_{12}a_{11} - a_{11} \circ d_{12}^*) \\ &\quad + \Phi(d_{12}b_{12} - b_{12} \circ d_{12}^*) \end{aligned}$$

which results that

$$\begin{aligned} \Phi(d_{12}c_{22} - \frac{1}{2}(b_{12}d_{12}^* + c_{22}d_{12}^* + d_{12}^*c_{11} + d_{12}^*b_{12})) &= \Phi(-\frac{1}{2}d_{12}^*a_{11}) \\ &\quad + \Phi(-b_{12} \circ d_{12}^*). \end{aligned}$$

It follows that

$$\begin{aligned} &\Phi(p_2(d_{12}c_{22} - \frac{1}{2}(b_{12}d_{12}^* + c_{22}d_{12}^* + d_{12}^*c_{11} + d_{12}^*b_{12})) \\ &\quad - (d_{12}c_{22} - \frac{1}{2}(b_{12}d_{12}^* + c_{22}d_{12}^* + d_{12}^*c_{11} + d_{12}^*b_{12})) \circ p_2^*) \\ &= \Phi(p_2(-\frac{1}{2}d_{12}^*a_{11}) - (-\frac{1}{2}d_{12}^*a_{11}) \circ p_2^*) + \Phi(p_2(-b_{12} \circ d_{12}^*) - (-b_{12} \circ d_{12}^*) \circ p_2^*) \end{aligned}$$

which leads to

$$\Phi(-\frac{1}{2}d_{12}c_{22} - \frac{1}{2^2}(c_{22}d_{12}^* + d_{12}^*c_{11})) = \Phi(-\frac{1}{2^2}d_{12}^*a_{11}).$$

This implies that $-\frac{1}{2}d_{12}c_{22} - \frac{1}{2^2}(c_{22}d_{12}^* + d_{12}^*c_{11}) = -\frac{1}{2^2}d_{12}^*a_{11}$ which results in $c_{11} = a_{11}$ and $c_{22} = 0$.

Likewise, we prove the cases (ii), (iii) and (iv). $\square$

Claim 2.2. For any elements $a_{12} \in \mathcal{A}_{12}$ and $b_{21} \in \mathcal{A}_{21}$ holds $\Phi(a_{12} + b_{21}) = \Phi(a_{12}) + \Phi(b_{21})$.

Proof. From supposition of Φ , consider an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ satisfying $\Phi(c) = \Phi(a_{12}) + \Phi(b_{21})$. For any element $d_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned} \Phi(d_{12}c - c \circ d_{12}^*) &= \Phi(d_{12}a_{12} - a_{12} \circ d_{12}^*) + \Phi(d_{12}b_{21} - b_{21} \circ d_{12}^*) \\ &= \Phi(-(a_{12} \circ d_{12}^*)) + \Phi(d_{12}b_{21}) \end{aligned}$$

which implies that

$$\begin{aligned} &\Phi(p_2(d_{12}c - c \circ d_{12}^*) - (d_{12}c - c \circ d_{12}^*) \circ p_2^*) \\ &= \Phi(p_2(-(a_{12} \circ d_{12}^*)) - (-(a_{12} \circ d_{12}^*)) \circ p_2^*) \end{aligned}$$

$$+ \Phi(p_2(d_{12}b_{21}) - (d_{12}b_{21}) \circ p_2^*) = 0.$$

This leads to $p_2(d_{12}c - c \circ d_{12}^*) - (d_{12}c - c \circ d_{12}^*) \circ p_2^* = 0$ that produces $c_{11} = c_{22} = 0$. As result, we obtain

$$\Phi(c_{12} + c_{21}) = \Phi(a_{12}) + \Phi(b_{21}).$$

It follows that, for any element $d_{12} \in \mathcal{A}_{12}$ we have

$$\begin{aligned} \Phi(d_{12}(c_{12} + c_{21}) - (c_{12} + c_{21}) \circ d_{12}^*) &= \Phi(d_{12}a_{12} - a_{12} \circ d_{12}^*) \\ &+ \Phi(d_{12}b_{21} - b_{21} \circ d_{12}^*) = \Phi(-(a_{12} \circ d_{12}^*)) + \Phi(d_{12}b_{21}). \end{aligned}$$

On account of this, we obtain

$$\begin{aligned} \Phi((ip_2)(d_{12}c_{21} - c_{12} \circ d_{12}^*) - (d_{12}c_{21} - c_{12} \circ d_{12}^*) \circ (ip_2)^*) \\ = \Phi((ip_2)(-(a_{12} \circ d_{12}^*)) - (-(a_{12} \circ d_{12}^*)) \circ (ip_2)^*) + \Phi((ip_2)(d_{12}b_{21}) \\ - (d_{12}b_{21}) \circ (ip_2)^*) = \Phi((ip_2)(-(a_{12} \circ d_{12}^*)) - (-(a_{12} \circ d_{12}^*)) \circ (ip_2)^*), \end{aligned}$$

which implies that

$$\begin{aligned} (ip_2)(d_{12}c_{21} - c_{12} \circ d_{12}^*) - (d_{12}c_{21} - c_{12} \circ d_{12}^*) \circ (ip_2)^* \\ = (ip_2)(-(a_{12} \circ d_{12}^*)) - (-(a_{12} \circ d_{12}^*)) \circ (ip_2)^* \end{aligned}$$

and resulting in $d_{12}^*c_{12} = d_{12}^*a_{12}$. It shows that $c_{12} = a_{12}$. Next, for any element $d_{21} \in \mathcal{A}_{21}$

$$\begin{aligned} \Phi(d_{21}(a_{12} + c_{21}) - (a_{12} + c_{21}) \circ d_{21}^*) &= \Phi(d_{21}a_{12} - a_{12} \circ d_{21}^*) \\ &+ \Phi(d_{21}b_{21} - b_{21} \circ d_{21}^*) = \Phi(d_{21}a_{12}) + \Phi(-(b_{21} \circ d_{21}^*)). \end{aligned}$$

As a results, we obtain

$$\begin{aligned} \Phi((ip_1)(d_{21}a_{12} - c_{21} \circ d_{21}^*) - (d_{21}a_{12} - c_{21} \circ d_{21}^*) \circ (ip_1)^*) \\ = \Phi((ip_1)(d_{21}a_{12}) - (d_{21}a_{12}) \circ (ip_1)^*) + \Phi((ip_1)(-(b_{21} \circ d_{21}^*)) \\ - (-(b_{21} \circ d_{21}^*)) \circ (ip_1)^*) = \Phi((ip_1)(-(b_{21} \circ d_{21}^*)) - (-(b_{21} \circ d_{21}^*)) \circ (ip_1)^*) \end{aligned}$$

which leads to

$$\begin{aligned} (ip_1)(d_{21}a_{12} - c_{21} \circ d_{21}^*) - (d_{21}a_{12} - c_{21} \circ d_{21}^*) \circ (ip_1)^* \\ = (ip_1)(-(b_{21} \circ d_{21}^*)) - (-(b_{21} \circ d_{21}^*)) \circ (ip_1)^*. \end{aligned}$$

This implies that $d_{21}^*c_{21} = d_{21}^*b_{21}$ which results in $c_{21} = b_{21}$. $\square$

Claim 2.3. For any elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$ and (ii) $\Phi(b_{12} + c_{21} + d_{22}) = \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.

Proof. Consider an element $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{A}$ verifying $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$. By Claim 2.2 we have

$$\begin{aligned} \Phi(p_2 f - f \circ p_2^*) &= \Phi(p_2 a_{11} - a_{11} \circ p_2^*) + \Phi(p_2 b_{12} - b_{12} \circ p_2^*) + \Phi(p_2 c_{21} - c_{21} \circ p_2^*) \\ &= \Phi(-\frac{1}{2}b_{12}) + \Phi(\frac{1}{2}c_{21}) = \Phi(-\frac{1}{2}b_{12} + \frac{1}{2}c_{21}) \end{aligned}$$

which implies that $p_2 f - f \circ p_2^* = -\frac{1}{2}b_{12} + \frac{1}{2}c_{21}$. As result we obtain $f_{12} = b_{12}$ and $f_{21} = c_{21}$. Hence,

$$\Phi(f_{11} + b_{12} + c_{21} + f_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}).$$

As a consequence, we have

$$\begin{aligned} &\Phi((ip_2)(f_{11} + b_{12} + c_{21} + f_{22}) - (f_{11} + b_{12} + c_{21} + f_{22}) \circ (ip_2)^*) \\ &= \Phi((ip_2)a_{11} - a_{11} \circ (ip_2)^*) + \Phi((ip_2)b_{12} - b_{12} \circ (ip_2)^*) + \Phi((ip_2)c_{21} - c_{21} \circ (ip_2)^*) \end{aligned}$$

which results that

$$\begin{aligned} &\Phi((ip_2)(f_{11} + b_{12} + c_{21} + f_{22}) - (f_{11} + b_{12} + c_{21} + f_{22}) \circ (ip_2)^*) \\ &= \Phi(\frac{1}{2}ib_{12}) + \Phi(\frac{3}{2}ic_{21}) = \Phi(i(\frac{1}{2}b_{12} + \frac{3}{2}c_{21})). \end{aligned}$$

This shows that $(ip_2)(f_{11} + b_{12} + c_{21} + f_{22}) - (f_{11} + b_{12} + c_{21} + f_{22}) \circ (ip_2)^* = i(\frac{1}{2}b_{12} + \frac{3}{2}c_{21})$ which yields $f_{22} = 0$. It follows that

$$\begin{aligned} &\Phi(d_{12}(f_{11} + b_{12} + c_{21}) - (f_{11} + b_{12} + c_{21}) \circ d_{12}^*) \\ &= \Phi(d_{12}a_{11} - a_{11} \circ d_{12}^*) + \Phi(d_{12}b_{12} - b_{12} \circ d_{12}^*) + \Phi(d_{12}c_{21} - c_{21} \circ d_{12}^*) \\ &= \Phi(-\frac{1}{2}d_{12}^*a_{11}) + \Phi(-(b_{12} \circ d_{12}^*)) + \Phi(d_{12}c_{21}) \end{aligned}$$

which implies that

$$\begin{aligned}
& \Phi(p_2(d_{12}c_{21} - \frac{1}{2}d_{12}^*f_{11} - b_{12} \circ d_{12}^*) - (d_{12}c_{21} - \frac{1}{2}d_{12}^*f_{11} - b_{12} \circ d_{12}^*) \circ p_2^*) \\
&= \Phi(p_2(-\frac{1}{2}d_{12}^*a_{11}) - (-\frac{1}{2}d_{12}^*a_{11}) \circ p_2^*) + \Phi(p_2(-(b_{12} \circ d_{12}^*)) \\
&\quad - (- (b_{12} \circ d_{12}^*)) \circ p_2^*) + \Phi(p_2(d_{12}c_{21}) - (d_{12}c_{21}) \circ p_2^*) \\
&= \Phi(p_2(-\frac{1}{2}d_{12}^*a_{11}) - (-\frac{1}{2}d_{12}^*a_{11}) \circ p_2^*).
\end{aligned}$$

On account of this, we obtain $d_{12}^*f_{11} = d_{12}^*a_{11}$ which implies that $f_{11} = a_{11}$.

Likewise, we prove the case (ii). $\square$

Claim 2.4. For any elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ holds $\Phi(a_{11} + b_{12} + c_{21} + d_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.

Proof. Consider an element $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{A}$ verifying $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$. By Claims 2.1(i) and 2.3(i) we have

$$\begin{aligned}
& \Phi((ip_1)f - f \circ (ip_1)^*) = \Phi((ip_1)a_{11} - a_{11} \circ (ip_1)^*) + \Phi((ip_1)b_{12} - b_{12} \circ (ip_1)^*) \\
&\quad + \Phi((ip_1)c_{21} - c_{21} \circ (ip_1)^*) + \Phi((ip_1)d_{22} - d_{22} \circ (ip_1)^*) \\
&= \Phi(2ia_{11}) + \Phi(\frac{3}{2}ib_{12}) + \Phi(\frac{1}{2}ic_{21}) = \Phi(i(2a_{11} + \frac{3}{2}b_{12} + \frac{1}{2}c_{21})).
\end{aligned}$$

It follows that $(ip_1)f - f \circ (ip_1)^* = i(2a_{11} + \frac{3}{2}b_{12} + \frac{1}{2}c_{21})$ which implies that $f_{11} = a_{11}$, $f_{12} = b_{12}$ and $f_{21} = c_{21}$. Also,

$$\begin{aligned}
& \Phi((ip_2)f - f \circ (ip_2)^*) = \Phi((ip_2)a_{11} - a_{11} \circ (ip_2)^*) + \Phi((ip_2)b_{12} - b_{12} \circ (ip_2)^*) \\
&\quad + \Phi((ip_2)c_{21} - c_{21} \circ (ip_2)^*) + \Phi((ip_2)d_{22} - d_{22} \circ (ip_2)^*) \\
&= \Phi(\frac{1}{2}ib_{12}) + \Phi(\frac{3}{2}ic_{21}) + \Phi(2id_{22}) = \Phi(i(\frac{1}{2}b_{12} + \frac{3}{2}c_{21} + 2d_{22})).
\end{aligned}$$

This results that $(ip_2)f - f \circ (ip_2)^* = i(\frac{1}{2}b_{12} + \frac{3}{2}c_{21} + 2d_{22})$ which implies that $f_{22} = d_{22}$. $\square$

Claim 2.5. For any elements $a_{12}, b_{12} \in \mathcal{A}_{12}$ and $a_{21}, b_{21} \in \mathcal{A}_{21}$ hold: (i) $\Phi(a_{12} + b_{12}) = \Phi(a_{12}) + \Phi(b_{12})$ and (ii) $\Phi(a_{21} + b_{21}) = \Phi(a_{21}) + \Phi(b_{21})$.

Proof. First, we note that the following identity is valid

$$\begin{aligned} (p_1 + a_{12})(p_2 + b_{12}) - (p_2 + b_{12}) \circ (p_1 + a_{12})^* \\ = a_{12} + \frac{1}{2}b_{12} - \frac{1}{2}a_{12}^* - \frac{1}{2}b_{12}a_{12}^* - \frac{1}{2}a_{12}^*b_{12}. \end{aligned}$$

On account of this, by Claim 2.4 we have

$$\begin{aligned} & \Phi(a_{12} + \frac{1}{2}b_{12}) + \Phi(-\frac{1}{2}a_{12}^*) + \Phi(-\frac{1}{2}b_{12}a_{12}^*) + \Phi(-\frac{1}{2}a_{12}^*b_{12}) \\ = & \Phi(a_{12} + \frac{1}{2}b_{12} - \frac{1}{2}a_{12}^* - \frac{1}{2}b_{12}a_{12}^* - \frac{1}{2}a_{12}^*b_{12}) \\ = & \Phi((p_1 + a_{12})(p_2 + b_{12}) - (p_2 + b_{12}) \circ (p_1 + a_{12})^*) \\ = & \Phi(p_1 + a_{12})\Phi(p_2 + b_{12}) - \Phi(p_2 + b_{12}) \circ \Phi(p_1 + a_{12})^* \\ = & (\Phi(p_1) + \Phi(a_{12}))(\Phi(p_2) + \Phi(b_{12})) \\ & - (\Phi(p_2) + \Phi(b_{12})) \circ (\Phi(p_1)^* + \Phi(a_{12})^*) \\ = & \Phi(p_1)\Phi(p_2) - \Phi(p_2) \circ \Phi(p_1)^* + \Phi(p_1)\Phi(b_{12}) - \Phi(b_{12}) \circ \Phi(p_1)^* \\ & + \Phi(a_{12})\Phi(p_2) - \Phi(p_2) \circ \Phi(a_{12})^* + \Phi(a_{12})\Phi(b_{12}) - \Phi(b_{12}) \circ \Phi(a_{12})^* \\ = & \Phi(p_1p_2 - p_2 \circ p_1^*) + \Phi(p_1b_{12} - b_{12} \circ p_1^*) + \Phi(a_{12}p_2 - p_2 \circ a_{12}^*) \\ & + \Phi(a_{12}b_{12} - b_{12} \circ a_{12}^*) \\ = & \Phi(\frac{1}{2}b_{12}) + \Phi(a_{12} - \frac{1}{2}a_{12}^*) + \Phi(-\frac{1}{2}b_{12}a_{12}^* - \frac{1}{2}a_{12}^*b_{12}) \\ = & \Phi(a_{12}) + \Phi(\frac{1}{2}b_{12}) + \Phi(-\frac{1}{2}a_{12}^*) + \Phi(-\frac{1}{2}b_{12}a_{12}^*) + \Phi(-\frac{1}{2}a_{12}^*b_{12}). \end{aligned}$$

This allows us to conclude that $\Phi(a_{12} + b_{12}) = \Phi(a_{12}) + \Phi(b_{12})$.

Likewise, we prove the case (ii) using the identity

$$\begin{aligned} (p_2 + a_{21})(p_1 + b_{21}) - (p_1 + b_{21}) \circ (p_2 + a_{21})^* \\ = a_{21} + \frac{1}{2}b_{21} - \frac{1}{2}a_{21}^* - \frac{1}{2}b_{21}a_{21}^* - \frac{1}{2}a_{21}^*b_{21}. \end{aligned}$$

□

Claim 2.6. For any elements $a_{11}, b_{11} \in \mathcal{A}_{11}$ and $a_{22}, b_{22} \in \mathcal{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{11}) = \Phi(a_{11}) + \Phi(b_{11})$ and (ii) $\Phi(a_{22} + b_{22}) = \Phi(a_{22}) + \Phi(b_{22})$.

Proof. Consider an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ verifying $\Phi(c) = \Phi(a_{11}) + \Phi(b_{11})$. By Claim 2.5(ii), for any element $d_{12} \in \mathcal{A}_{12}$ we have

$$\begin{aligned}\Phi(d_{12}c - c \circ d_{12}^*) &= \Phi(d_{12}a_{11} - a_{11} \circ d_{12}^*) + \Phi(d_{12}b_{11} - b_{11} \circ d_{12}^*) \\ &= \Phi(-\frac{1}{2}d_{12}^*a_{11}) + \Phi(-\frac{1}{2}d_{12}^*b_{11}) = \Phi(-\frac{1}{2}d_{12}^*(a_{11} + b_{11})).\end{aligned}$$

This results that $d_{12}c - c \circ d_{12}^* = -\frac{1}{2}d_{12}^*(a_{11} + b_{11})$ which leads to $c_{11} = a_{11} + b_{11}$ and $c_{12} = c_{21} = c_{22} = 0$.

Likewise, we prove the case (ii). $\square$

Claim 2.7. Φ is an additive mapping.

Proof. The result is an immediate consequence of Claims 2.4, 2.5 and 2.6. $\square$

Lemma 2.4. Let $\mathcal{A}$ be a factor von Neumann algebra with identity $1_{\mathcal{A}}$ and an element $a \in \mathcal{A}$. If $\mathcal{A}$ satisfies the condition: $ab - b \circ a^* = 0$, for all elements $b \in \mathcal{A}$, then $a \in \mathbb{R}1_{\mathcal{A}}$.

Proof. Note that, when we take $b = 1$, the condition reduces to $a - a^* = 0$. As a consequence, we obtain $ab - ba = 0$, for all elements $b \in \mathcal{A}$. This means that a belongs to the center of $\mathcal{A}$ which leads to $a \in \mathbb{R}1_{\mathcal{A}}$. $\square$

In the remainder of this paper, all lemmas satisfy the conditions of the Main Theorem.

Lemma 2.5. $\Phi(\mathbb{R}1_{\mathcal{A}}) = \mathbb{R}1_{\mathcal{B}}$, the mapping Φ preserves Hermitian elements in both directions and $\Phi(\mathbb{C}1_{\mathcal{A}}) = \mathbb{C}1_{\mathcal{B}}$.

Proof. First, observe that for any elements $a, b \in \mathcal{A}$ holds $ab - b \circ a^* = 0$ if and only if $\Phi(a)\Phi(b) - \Phi(a) \circ \Phi(b)^* = 0$. Hence, for any elements $\alpha \in \mathbb{R}$ and $b \in \mathcal{A}$, $0 = \Phi((\alpha 1_{\mathcal{A}})b - b \circ (\alpha 1_{\mathcal{A}})^*) = \Phi(\alpha 1_{\mathcal{A}})\Phi(b) - \Phi(b) \circ \Phi(\alpha 1_{\mathcal{A}})^* = 0$ which implies that $\Phi(\alpha 1_{\mathcal{A}}) \in \mathbb{R}1_{\mathcal{B}}$, by Lemma 2.4. As α is an arbitrary element, we conclude that $\Phi(\mathbb{R}1_{\mathcal{A}}) \subseteq \mathbb{R}1_{\mathcal{B}}$. In view of the fact that Φ^{-1} has the same properties of Φ , then by a similar argument we prove that $\mathbb{R}1_{\mathcal{B}} \subseteq \Phi(\mathbb{R}1_{\mathcal{A}})$. As a consequence, we obtain $\Phi(\mathbb{R}1_{\mathcal{A}}) = \mathbb{R}1_{\mathcal{B}}$.

Let a be an element of $\mathcal{A}$ satisfying the condition $a = a^*$. Then $0 = \Phi(a 1_{\mathcal{A}} - 1_{\mathcal{A}} \circ a^*) = \Phi(a)\Phi(1_{\mathcal{A}}) - \Phi(1_{\mathcal{A}}) \circ \Phi(a)^*$ which implies that $\Phi(a) = \Phi(a)^*$. In view of the fact that Φ^{-1} has the same properties of Φ , then by a similar argument we prove that if $\Phi(a) = \Phi(a)^*$, then $a = a^*$, for all such

elements $a \in \mathcal{A}$. Therefore, $a = a^*$ if and only if $\Phi(a) = \Phi(a)^*$, for all elements $a \in \mathcal{A}$.

For any elements $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$ verifying the condition $a = a^*$, we have $0 = \Phi(a(\lambda 1_{\mathcal{A}}) - (\lambda 1_{\mathcal{A}}) \circ a^*) = \Phi(a)\Phi(\lambda 1_{\mathcal{A}}) - \Phi(\lambda 1_{\mathcal{A}}) \circ \Phi(a)^*$ which implies that $\Phi(a)\Phi(\lambda 1_{\mathcal{A}}) - \Phi(\lambda 1_{\mathcal{A}})\Phi(a) = 0$. This results that $b\Phi(\lambda 1_{\mathcal{A}}) - \Phi(\lambda 1_{\mathcal{A}})b = 0$, for any element $b \in \mathcal{B}$ verifying $b = b^*$. As a result, we obtain $b\Phi(\lambda 1_{\mathcal{A}}) - \Phi(\lambda 1_{\mathcal{A}})b = 0$, for any element $b \in \mathcal{B}$, since any element $b \in \mathcal{B}$ can be written as $b = b_1 + ib_2$, where $b_1 = \frac{b+b^*}{2}$ and $b_2 = \frac{b-b^*}{2i}$. This shows that $\Phi(\lambda 1_{\mathcal{A}}) \in \mathbb{C}1_{\mathcal{B}}$. As λ is any element, we conclude that $\Phi(\mathbb{C}1_{\mathcal{A}}) \subseteq \mathbb{C}1_{\mathcal{B}}$. Applying a similar argument to mapping Φ^{-1} we obtain $\mathbb{C}1_{\mathcal{B}} \subseteq \Phi(\mathbb{C}1_{\mathcal{A}})$. This means that $\Phi(\mathbb{C}1_{\mathcal{A}}) = \mathbb{C}1_{\mathcal{B}}$. $\square$

Lemma 2.6. *Either $\Phi(ia) = i\Phi(a)$, for all elements $a \in \mathcal{A}$, or $\Phi(ia) = -i\Phi(a)$, for all elements $a \in \mathcal{A}$.*

Proof. From supposition of Φ and by Lemma 2.5, we have $\Phi(\pm \frac{1}{2}i1_{\mathcal{A}}) \in (\mathbb{C} \setminus \mathbb{R})1_{\mathcal{B}}$ and $\Phi(\pm \frac{1}{2}1_{\mathcal{A}}) \in \mathbb{R}1_{\mathcal{B}}$. As $\pm \frac{1}{2}i1_{\mathcal{A}} = (\pm \frac{1}{2}i1_{\mathcal{A}})(\frac{1}{2}1_{\mathcal{A}}) - (\frac{1}{2}1_{\mathcal{A}}) \circ (\pm \frac{1}{2}i1_{\mathcal{A}})^*$, then

$$\begin{aligned} \Phi(\pm \frac{1}{2}i1_{\mathcal{A}}) &= \Phi((\pm \frac{1}{2}i1_{\mathcal{A}})(\frac{1}{2}1_{\mathcal{A}}) - (\frac{1}{2}1_{\mathcal{A}}) \circ (\pm \frac{1}{2}i1_{\mathcal{A}})^*) \\ &= \Phi(\pm \frac{1}{2}i1_{\mathcal{A}})\Phi(\frac{1}{2}1_{\mathcal{A}}) - \Phi(\frac{1}{2}1_{\mathcal{A}}) \circ \Phi(\pm \frac{1}{2}i1_{\mathcal{A}})^* \end{aligned}$$

which implies that $\Phi(\pm \frac{1}{2}i1_{\mathcal{A}})^* = (1_{\mathcal{A}} - \Phi(\frac{1}{2}1_{\mathcal{A}})^{-1})\Phi(\pm \frac{1}{2}i1_{\mathcal{A}})$. This results that $\Phi(\pm \frac{1}{2}i1_{\mathcal{A}}) \in \mathbb{R}i1_{\mathcal{B}}$. As consequence we have

$$\begin{aligned} \Phi(\frac{1}{2}i1_{\mathcal{A}}) &= \Phi((-\frac{1}{2}i1_{\mathcal{A}})(-\frac{1}{2}1_{\mathcal{A}}) - (-\frac{1}{2}1_{\mathcal{A}}) \circ (-\frac{1}{2}i1_{\mathcal{A}})^*) \\ &= \Phi(-\frac{1}{2}i1_{\mathcal{A}})\Phi(-\frac{1}{2}1_{\mathcal{A}}) - \Phi(-\frac{1}{2}1_{\mathcal{A}}) \circ \Phi(-\frac{1}{2}i1_{\mathcal{A}})^* \\ &= 2\Phi(-\frac{1}{2}1_{\mathcal{A}})\Phi(-\frac{1}{2}i1_{\mathcal{A}}), \quad (2) \end{aligned}$$

since $\Phi(\pm \frac{1}{2}i1_{\mathcal{A}})^* = -\Phi(\pm \frac{1}{2}i1_{\mathcal{A}})$, and

$$\Phi(-\frac{1}{2}1_{\mathcal{A}}) = \Phi((-\frac{1}{2}i1_{\mathcal{A}})(-\frac{1}{2}i1_{\mathcal{A}}) - (-\frac{1}{2}i1_{\mathcal{A}}) \circ (-\frac{1}{2}i1_{\mathcal{A}})^*)$$

$$= \Phi(-\frac{1}{2}i1_{\mathcal{A}})\Phi(-\frac{1}{2}i1_{\mathcal{A}}) - \Phi(-\frac{1}{2}i1_{\mathcal{A}}) \circ \Phi(-\frac{1}{2}i1_{\mathcal{A}})^* = 2\Phi(-\frac{1}{2}i1_{\mathcal{A}})^2. \quad (3)$$

By Claim 2.7 and (2)-(3) we conclude that $\Phi(-\frac{1}{2}1_{\mathcal{A}}) = -\frac{1}{2}1_{\mathcal{A}}$ and $\Phi(\frac{1}{2}i1_{\mathcal{A}}) = \pm\frac{1}{2}i1_{\mathcal{A}}$. Thus, for any element $a \in \mathcal{A}$, we have that $\Phi(ia) = \Phi((\frac{1}{2}i1_{\mathcal{A}}) \circ a - a(\frac{1}{2}i1_{\mathcal{A}})^*) = \Phi(\frac{1}{2}i1_{\mathcal{A}}) \circ \Phi(a) - \Phi(a)\Phi(\frac{1}{2}i1_{\mathcal{A}})^* = (\Phi(\frac{1}{2}i1_{\mathcal{A}}) - \Phi(\frac{1}{2}i1_{\mathcal{A}})^*)\Phi(a) = 2\Phi(\frac{1}{2}i1_{\mathcal{A}})\Phi(a)$. This means that, either $\Phi(ia) = i\Phi(a)$, for all elements $a \in \mathcal{A}$, or $\Phi(ia) = -i\Phi(a)$, for all elements $a \in \mathcal{A}$. $\square$

Lemma 2.7. $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -ring isomorphism.

Proof. Two cases are considered. First case: $\Phi(ia) = i\Phi(a)$, for all elements $a \in \mathcal{A}$. For any elements $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \Phi(ab + b \circ a^*) &= \Phi((ia)(-ib) - (-ib) \circ (ia)^*) = \Phi(ia)\Phi(-ib) \\ &\quad - \Phi(-ib) \circ \Phi(ia)^* = (i\Phi(a))(-i\Phi(b)) - (-i\Phi(b)) \circ (i\Phi(a))^* \\ &= \Phi(a)\Phi(b) + \Phi(b) \circ \Phi(a)^*. \end{aligned} \quad (4)$$

Taking into account (1) and (4) we get $\Phi(ab) = \Phi(a)\Phi(b)$ and $\Phi(b \circ a^*) = \Phi(b) \circ \Phi(a)^*$. It follows that $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and, for any element $a \in \mathcal{A}$, $\Phi(a^*) = \Phi(1_{\mathcal{A}} \circ a^*) = \Phi(1_{\mathcal{A}}) \circ \Phi(a^*)^* = 1_{\mathcal{B}} \circ \Phi(a^*)^* = \Phi(a)^*$. As result we obtain that Φ is a $*$ -ring isomorphism. Second case: $\Phi(ia) = -i\Phi(a)$, for all elements $a \in \mathcal{A}$. The proof is entirely similar as in the first case.

The Theorem is proved. $\square$

Corollary 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be type I factor von Neumann algebras acting on complex Hilbert spaces H and K , respectively. Then every bijective mapping $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ satisfies $\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*$, for all elements a, b of $\mathcal{A}$, if and only if there is a unitary or conjugate unitary operator $U : H \rightarrow K$ such that $\phi(a) = UaU^*$, for all elements $a \in \mathcal{A}$.*

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**SOME RESULTS ON WIENER INDEX, HYPER-WIENER
INDEX AND ZAGREB INDEX**

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Abstract

A topological index is a real number derived from the structure of a graph, which is invariant under graph isomorphism. Many topological indices are closely correlated with some physico-chemical characteristics of the underlying compounds. In this paper, some topological indices such as Wiener index, hyper-Wiener index and Zagreb index of a generalized join of some certain graphs are obtained.

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1 Introduction

Let $G = (V, E)$ be a simple connected graph and $V(G)$ and $E(G)$ denote the vertex set and edge set of G , respectively. For a vertex v of G , $N_G(v)$ denotes the set of vertices of G that are adjacent to v in G , and we denote $|N_G(v)|$ by $d_G(v)$. For distinct vertices u and v of G , we write $u \sim v$ if u and v are adjacent in G and the edge e between u and v will be denoted by $e = uv$. Also the *distance* between two distinct vertices u and v in G , denoted by $d_G(u, v)$, is the length of the shortest path connecting u and v , if such a path exists; otherwise, we set $d_G(u, v) := \infty$. A *topological index* is a real number derived from the structure of a graph, which is invariant under graph isomorphism. Many topological indices are closely correlated with some physico-chemical characteristics of the underlying compounds. The first and second *Zagreb indices* of a graph denoted by $M_1(G)$ and $M_2(G)$, respectively, are degree based topological indices introduced more than thirty years ago by I. Gutman and N. Trinajstić [8]. They are defined as

$$M_1(G) = \sum_{e=uv \in E(G)} (d_G(u) + d_G(v)) = \sum_{v \in V(G)} d_G^2(v),$$

$$M_2(G) = \sum_{e=uv \in E(G)} d_G(u)d_G(v).$$

These indices were introduced to study the structure-dependency of the total π -electron energy (ε). It was found that ε depends on $M_1(G)$ and thus provides a measure of carbon skeleton of the underlying molecules.

The *Hyper Zagreb index* of a graph denoted by $HM(G)$, was introduced in [12], as a new version of Zagreb index. For a graph G , the Hyper Zagreb index is defined as follows:

$$HM(G) = \sum_{e=uv \in E(G)} (d_G(u) + d_G(v))^2.$$

In [13], the authors considered the multiplicative variants of molecular structure. When this idea is applied to Zagreb indices, one arrives at their

multiplicative versions $\Pi_1(G)$ and $\Pi_2(G)$, defined as

$$\Pi_1(G) = \prod_{v \in V(G)} d^2(v) \text{ and } \Pi_2(G) = \prod_{uv \in E(G)} d(u)d(v).$$

Finally, Eliasi et al. [4] defined a multiplicative version of M_1 as

$$\Pi_1^*(G) = \prod_{uv \in E(G)} (d(u) + d(v)),$$

and is called as the *multiplicative sum Zagreb index* by Xu and Das [15].

The *Wiener index*, $W(G)$, is equal to the count of all shortest distances in a graph (cf. [14]). In other words, $W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)$. Wiener number was defined in 1947 by an American chemist H. Wiener. He used this index to estimate the boiling point of Alkane. There are many situations in communication, facility location, cryptology, architecture etc. where the Wiener index of the corresponding graph or the average distance is of great interest. One of these problems, for example, is to find a spanning tree with minimum average distance. The Wiener index is one of the most studied topological indices, both from a theoretical point of view and applications, see for details [3], [5], [7] and [16].

The *hyper-Wiener index* of acyclic graphs was introduced by Milan Randić in 1993. Then Klein et al. [10], generalized Randić's definition for all connected graphs, as a generalization of the Wiener index. It is $WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d^2(u, v)$, where $d^2(u, v) = d(u, v)^2$. We encourage the reader to consult [1], [2], [6] and [9] for the mathematical properties of hyper-Wiener index and its applications in chemistry.

In Section 2 of the paper, we study Wiener index and hyper-Wiener index of a generalized join of some certain graphs. In Section 3, the first and second Zagreb indices and their multiplicative versions of a generalized join of some certain graphs are investigated; and also the Hyper Zagreb index is obtained.

All graphs considered in this paper are connected and simple. We say that G is an *empty graph* if $E(G) = \emptyset$. Also, K_n and $\overline{K_n}$ denote the complete graph with n vertices and its complement, respectively, and P_n denotes the path with n vertices.

2 Wiener index and hyper-Wiener index

In this section, first we recall the definition of a generalized join of graphs. For two graphs H_1 and H_2 with disjoint vertex sets, the *join* $H_1 \vee H_2$ of the graphs H_1 and H_2 is the graph obtained from the union of H_1 and H_2 by adding new edges from each vertex of H_1 to every vertex of H_2 . The concept of join graph is generalized (in [11], it is called as a generalized composition graph). Let G be a graph on k vertices with $V(G) = \{v_1, v_2, \dots, v_k\}$, and let $H_1, H_2, \dots, H_k$ be k pairwise disjoint graphs. The G -generalized join graph $G[H_1, H_2, \dots, H_k]$ of $H_1, H_2, \dots, H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to each vertex of H_j whenever $v_i \sim v_j$ in the graph G . Now, if the graph G consists of two adjacent vertices, then the G -generalized join graph $G[H_1, H_2]$ coincides with the join $H_1 \vee H_2$ of the graphs H_1 and H_2 .

In the rest of the paper, in the G -generalized join graph $G[H_1, H_2, \dots, H_k]$, we assume that each H_i is either a complete graph or it is an empty graph, and without loss of generality, we assume that $H_1, \dots, H_t$ are complete graphs and $H_{t+1}, \dots, H_k$ are empty graphs, where $0 \leq t \leq k$. Also, we always assume that there exists $1 \leq i \leq k$ such that $|H_i| > 1$.

In the following theorem, we study the Wiener index and hyper-Wiener index of $G' = G[H_1, H_2, \dots, H_k]$, where G is isomorphic to the path P_k .

Theorem 2.1. *Assume that $G' = P_k[H_1, H_2, \dots, H_k]$, where $k > 1$. Then*

$$W(G') = \frac{1}{2} \sum_{i=1}^k |H_i| \Delta_{H_i}$$

and

$$WW(G') = \frac{1}{4} \sum_{i=1}^k |H_i| \Delta_{H_i} + \frac{1}{4} \sum_{i=1}^k |H_i| \Delta'_{H_i},$$

where for $1 \leq i \leq t$, we have

$$\Delta_{H_i} = |H_i| - 1 + \sum_{j=1}^k |i - j| |H_j| \quad \text{and} \quad \Delta'_{H_i} = |H_i| - 1 + \sum_{j=1}^k (i - j)^2 |H_j|,$$

and for $t + 1 \leq i \leq k$, we have

$$\Delta_{H_i} = 2(|H_i| - 1) + \sum_{j=1}^k |i - j| |H_j| \quad \text{and} \quad \Delta'_{H_i} = 4(|H_i| - 1) + \sum_{j=1}^k (i - j)^2 |H_j|.$$

Proof. Assume that $h \in H_i$, for some $1 \leq i \leq k$. Then we have

$$\begin{aligned} \sum_{x \in V(G')} d(h, x) &= \sum_{x \in H_i} d(h, x) + \sum_{x \in V(G') \setminus H_i} d(h, x) \\ &= \sum_{x \in H_i} d(h, x) + \sum_{j=1}^k |i - j| |H_j| \\ &= \begin{cases} n - 1 + \sum_{j=1}^k |i - j| |H_j| & H_i \cong K_n \\ 2(n - 1) + \sum_{j=1}^k |i - j| |H_j| & H_i \cong \overline{K}_n. \end{cases} \end{aligned}$$

It is easy to see that for each $h, h' \in H_i$, $\sum_{x \in V(G')} d(h, x) = \sum_{x \in V(G')} d(h', x)$. Let $\Delta_{H_i} = \sum_{x \in V(G')} d(h, x)$, for some $h \in H_i$. Thus we have

$$\sum_{h \in H_i} \sum_{x \in V(G')} d(h, x) = |H_i| \Delta_{H_i}.$$

Therefore

$$W(G') = \frac{1}{2} \sum_{i=1}^k |H_i| \Delta_{H_i}.$$

Now let $h \in H_i$, for some $1 \leq i \leq k$. Then we have

$$\begin{aligned} \sum_{x \in V(G')} d^2(h, x) &= \sum_{x \in H_i} d^2(h, x) + \sum_{x \in V(G') \setminus H_i} d^2(h, x) \\ &= \sum_{x \in H_i} d^2(h, x) + \sum_{j=1}^k (i - j)^2 |H_j| \\ &= \begin{cases} n - 1 + \sum_{j=1}^k (i - j)^2 |H_j| & H_i \cong K_n \\ 4(n - 1) + \sum_{j=1}^k (i - j)^2 |H_j| & H_i \cong \overline{K}_n. \end{cases} \end{aligned}$$

Clearly for each $h, h' \in H_i$, $\sum_{x \in V(G')} d^2(h, x) = \sum_{x \in V(G')} d^2(h', x)$. Let $\Delta'_{H_i} = \sum_{x \in V(G')} d^2(h, x)$, for some $h \in H_i$. Thus we have

$$\sum_{h \in H_i} \sum_{x \in V(G')} d^2(h, x) = |H_i| \Delta'_{H_i}.$$

Therefore

$$WW(G') = \frac{1}{4} \sum_{i=1}^k |H_i| \Delta_{H_i} + \frac{1}{4} \sum_{i=1}^k |H_i| \Delta'_{H_i}.$$

□

We end this section with the following theorem which determines the Wiener index and hyper-Wiener index of $G' = G[H_1, H_2, \dots, H_n]$, where G is isomorphic to the complete graph K_n .

Theorem 2.2. *Let $G' = K_n[H_1, H_2, \dots, H_n]$, where $n > 1$. Then*

$$W(G') = \frac{1}{2} \sum_{i=1}^n |H_i| \Delta_{H_i}$$

and

$$WW(G') = \frac{1}{4} \sum_{i=1}^n |H_i| \Delta_{H_i} + \frac{1}{4} \sum_{i=1}^n |H_i| \Delta'_{H_i},$$

where for $1 \leq i \leq n$,

$$\Delta_{H_i} = \begin{cases} t-1 + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong K_t \\ 2(t-1) + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong \overline{K}_t \end{cases}$$

and

$$\Delta'_{H_i} = \begin{cases} t-1 + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong K_t \\ 4(t-1) + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong \overline{K}_t. \end{cases}$$

Proof. Let $h, h' \in H_i$, for some $1 \leq i \leq n$. Then we have

$$\sum_{x \in V(G')} d(h, x) = \sum_{x \in H_i} d(h, x) + \sum_{x \in V(G') \setminus H_i} 1$$

$$\begin{aligned}
&= \sum_{x \in H_i} d(h, x) + \sum_{j=1, j \neq i}^n |H_j| \\
&= \begin{cases} t-1 + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong K_t \\ 2(t-1) + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong \overline{K}_t \end{cases} \\
&= \sum_{x \in V(G')} d(h', x)
\end{aligned}$$

Let $\Delta_{H_i} = \sum_{x \in V(G')} d(h, x)$, for some $h \in H_i$. Thus we have

$$\sum_{h \in H_i} \sum_{x \in V(G')} d(h, x) = |H_i| \Delta_{H_i}.$$

Therefore

$$W(G') = \frac{1}{2} \sum_{i=1}^n |H_i| \Delta_{H_i}.$$

Now, for $h, h' \in H_i$, where $1 \leq i \leq n$, we have

$$\begin{aligned}
\sum_{x \in V(G')} d^2(h, x) &= \sum_{x \in H_i} d^2(h, x) + \sum_{x \in V(G') \setminus H_i} 1 \\
&= \sum_{x \in H_i} d^2(h, x) + \sum_{j=1, j \neq i}^n |H_j| \\
&= \begin{cases} t-1 + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong K_t \\ 4(t-1) + \sum_{j=1, j \neq i}^n |H_j| & H_i \cong \overline{K}_t \end{cases} \\
&= \sum_{x \in V(G')} d^2(h', x)
\end{aligned}$$

Let $\Delta'_{H_i} = \sum_{x \in V(G')} d^2(h, x)$, for some $h \in H_i$. Thus we have

$$\sum_{h \in H_i} \sum_{x \in V(G')} d^2(h, x) = |H_i| \Delta'_{H_i}.$$

Therefore

$$WW(G') = \frac{1}{4} \sum_{i=1}^n |H_i| \Delta_{H_i} + \frac{1}{4} \sum_{i=1}^n |H_i| \Delta'_{H_i}.$$

□

3 Zagreb index

In this section, we investigate the first and second Zagreb indices and their multiplicative versions, and also the Hayper Zagreb index of $G[H_1, H_2, \dots, H_k]$. Since each H_i is either a complete graph or it is an empty graph, without loss of generality, we assume that $H_1, \dots, H_t$ are complete graphs and $H_{t+1}, \dots, H_k$ are empty graphs, where $0 \leq t \leq k$.

In the following theorem, we study the first and second Zagreb indices of $G[H_1, H_2, \dots, H_k]$.

Theorem 3.1. *Let $G' = G[H_1, H_2, \dots, H_k]$, where $k > 1$. Then*

$$M_1(G') = \sum_{i=1}^k |H_i| d_{G'}^2(v_i)$$

and

$$M_2(G') = \sum_{e=v_i v_j \in E(G)} |H_i| |H_j| d_{G'}(v_i) d_{G'}(v_j) + \sum_{i=1}^t \frac{|H_i|(|H_i| - 1)}{2} d_{G'}^2(v_i).$$

Proof. For each two distinct vertices x and y in H_i , where $1 \leq i \leq k$, we have $d_{G'}(x) = d_{G'}(y)$. So we have $\sum_{x \in H_i} d_{G'}^2(x) = |H_i| d_{G'}^2(v_i)$. Therefore we have

$$M_1(G') = \sum_{v \in V(G')} d_{G'}^2(v) = \sum_{i=1}^k |H_i| d_{G'}^2(v_i).$$

Note that for each edge $e = v_i v_j \in E(G)$, we have $|H_i| |H_j|$ edges in G' . Also $H_1, \dots, H_t$ are complete graphs, where $0 \leq t \leq k$. Hence the second Zagreb index of G' reads as follows:

$$M_2(G') = \sum_{e=v_i v_j \in E(G)} |H_i| |H_j| d_{G'}(v_i) d_{G'}(v_j) + \sum_{i=1}^t \frac{|H_i|(|H_i| - 1)}{2} d_{G'}^2(v_i).$$

□

The following proposition, which determines the Hyper Zagreb index of $G' = G[H_1, H_2, \dots, H_k]$, is obtained similar to the second Zagreb index.

Proposition 3.2. *The Hyper Zagreb index of $G' = G[H_1, H_2, \dots, H_k]$ is*

$$HM(G') = \sum_{e=v_i v_j \in E(G)} |H_i| |H_j| (d_{G'}(v_i) + d_{G'}(v_j))^2 + \sum_{i=1}^t 2 |H_i| (|H_i| - 1) d_{G'}^2(v_i).$$

We end this section with the following corollary which is about the multiplicative versions of Zagreb indices.

Corollary 3.3. *Assume that $G' = G[H_1, H_2, \dots, H_k]$, where $k > 1$. Then the following holds:*

- (i) $\Pi_1(G') = \prod_{i=1}^k (d_{G'}(v_i))^{2|H_i|}$.
- (ii) $\Pi_2(G') = \prod_{e=v_i v_j \in E(G)} (d_{G'}(v_i))^{|H_i|} (d_{G'}(v_j))^{|H_j|} \prod_{i=1}^t d_{G'}(v_i)^{|H_i|(|H_i|-1)}$.
- (iii) $\Pi_1^*(G') = \prod_{e=v_i v_j \in E(G)} (d_{G'}(v_i) + d_{G'}(v_j))^{|H_i||H_j|} \prod_{i=1}^t (2d_{G'}(v_i))^{\frac{|H_i|(|H_i|-1)}{2}}$.

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STUDY OF A TYPE OF GENERALIZED BASIC
HYPERGEOMETRIC SERIES

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Abstract

The aim of this paper is to examine the generalized basic hypergeometric series defined by

$$f(x) = {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, \dots, q^{b_r+m_r} \\ q^{b_1}, q^{b_2}, \dots, q^{b_r} \end{matrix} ; q, q^n x \right]$$

$M, m_1, m_2, \dots, m_r, n \in \mathbb{N}$, $b_1, b_2, \dots, b_r \neq 0, -1, -2, \dots$, $0 < q < 1$, for this, the effects of the parameters b_j, m_j ($j = 1, 2, \dots, r$), q and n on the function f are studied, considering some examples and establishing for each one the following characteristics of $f(x)$: the polynomial that defines it, the roots of same and its set of positivity. Also, various graphics which show the behavior of the function are included.

Keywords: Generalized basic hypergeometric series; polynomial; set of positivity; graphic representation.

AMS Subject Classification: 26A33.

1 Introduction

The aim of this paper is to examine the generalized basic hypergeometric series defined by

$$f(x) = {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, \dots, q^{b_r+m_r} \\ q^{b_1}, q^{b_2}, \dots, q^{b_r} \end{matrix} ; q, q^n x \right]$$

$M, m_1, m_2, \dots, m_r, n \in \mathbb{N}, b_1, b_2, \dots, b_r \neq 0, -1, -2, \dots, 0 < q < 1,$

for this, the effects of the parameters b_j, m_j ($j = 1, 2, \dots, r$), q and n on the function f are studied, considering some examples and establishing for each one the following characteristics of $f(x)$: the polynomial that defines it, the roots of same and its set of positivity. Also, various graphics which show the behavior of the function are included.

The results of this study will be used for establish some q -integral inequalities applying the fractional q -integral operator $L_q^n(\cdot)$ [4], which is defined in the following form:

$$L_q^n \{M, b_1, b_2, \dots, b_r, \gamma, m_1, m_2, \dots, m_r; f(x)\} =$$

$$\frac{x^{-\gamma-1}}{\Gamma_q(M+1)} \int_0^x t^\gamma {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \dots, q^{b_r+m_r} \\ q^{b_1}, \dots, q^{b_r} \end{matrix} ; q, q^n \frac{t}{x} \right] f(t) d_q t,$$

$M, m_1, m_2, \dots, m_r \in \mathbb{N}_0, n \in \mathbb{N}, \gamma \in \mathbb{C},$

$b_1, b_2, \dots, b_r \neq 0, -1, -2, \dots, 0 < q < 1, \left| \frac{t}{x} \right| < 1.$

The inequality technique is one of the very useful tools in the study of special functions and theory of approximations [1]. By applying fractional q -integral operators, many researchers have obtained a lot of fractional q -integral inequalities and applications see, for instance, [1], [2], [6]-[8].

Now we present some definitions necessary for the development of the next sections.

1.1 The q -shifted factorial

It is defined as: [5, p. 3, No. (1.2.15)]

$$(a; q)_n = \begin{cases} 1, & n = 0. \\ (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1}), & n = 1, 2, \dots \end{cases} \quad (1)$$

1.2 Basic hypergeometric series

This series was introduced by Heine and it is defined as: [5]

$$\phi(a, b; c; q, z) = {}_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n, \quad (2)$$

where it is assumed that $c \neq q^{-m}$ for $m = 0, 1, \dots$, and $(a; q)_n$ is the q -shifted factorial defined in (1).

1.3 Generalized basic hypergeometric series

A generalization of the basic hypergeometric series ${}_2\phi_1$, is given by: [5]

$${}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \quad (3)$$

where $b_1, \dots, b_s \neq q^{-m}$ for $m = 0, 1, 2, \dots$; $\binom{n}{2} = \frac{n(n-1)}{2}$; $q \neq 0$ when $r > s + 1$ and $\lim_{q \rightarrow 1} {}_r\phi_s = {}_rF_s$.

As particular case of (3) we have

$${}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, \dots, q^{b_r+m_r} \\ q^{b_1}, q^{b_2}, \dots, q^{b_r} \end{matrix} ; q, q^n x \right] = \sum_{k=0}^M \frac{(q^{-M}; q)_k (q^{b_1+m_1}; q)_k \cdots (q^{b_r+m_r}; q)_k}{(q; q)_k (q^{b_1}; q)_k \cdots (q^{b_r}; q)_k} (q^n x)^k, \quad (4)$$

$M, m_1, m_2, \dots, m_r, n \in \mathbb{N}, b_1, b_2, \dots, b_r \neq 0, -1, -2, \dots, 0 < q < 1$,
where we use the result [5, p. 4, No. (1.2.23)]

$$(q^{-m}; q)_n = 0 \quad n = m+1, m+2, \dots, \text{ with } m = 0, 1, 2, \dots, \text{ and } q \neq 0,$$

then the q -hypergeometric series given in (4) is terminating, which is in fact a polynomial of x [3].

Let be $f(x) \equiv {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \dots, q^{b_r+m_r} \\ q^{b_1}, \dots, q^{b_r} \end{matrix} ; q, q^n x \right]$. From (4) applying the definition (1) we can write

$$\begin{aligned} f(x) \equiv {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \dots, q^{b_r+m_r} \\ q^{b_1}, \dots, q^{b_r} \end{matrix} ; q, q^n x \right] = 1 + \\ \frac{(1 - q^{-M})(1 - q^{b_1+m_1}) \dots (1 - q^{b_r+m_r})}{(1 - q)(1 - q^{b_1}) \dots (1 - q^{b_r})} (q^n x) + \\ \frac{(1 - q^{-M})(1 - q^{-M+1})(1 - q^{b_1+m_1})(1 - q^{b_1+m_1+1})}{(1 - q)(1 - q^2)(1 - q^{b_1})(1 - q^{b_1+1})} \dots \times \\ \frac{(1 - q^{b_r+m_r})(1 - q^{b_r+m_r+1})}{(1 - q^{b_r})(1 - q^{b_r+1})} (q^n x)^2 + \dots + \\ \frac{(1 - q^{-M})(1 - q^{-M+1}) \dots (1 - q^{-1})}{(1 - q)(1 - q^2) \dots (1 - q^M)} \times \\ \frac{(1 - q^{b_1+m_1})(1 - q^{b_1+m_1+1}) \dots (1 - q^{b_1+m_1+M-1})}{(1 - q^{b_1})(1 - q^{b_1+1}) \dots (1 - q^{b_1+M-1})} \dots \times \\ \frac{(1 - q^{b_r+m_r})(1 - q^{b_r+m_r+1}) \dots (1 - q^{b_r+m_r+M-1})}{(1 - q^{b_r})(1 - q^{b_r+1}) \dots (1 - q^{b_r+M-1})} (q^n x)^M. \quad (5) \end{aligned}$$

The following results will be very useful for determine the sign of the factors that appear in (5).

$$(1 - q^A) > 0, \quad \text{if } 0 < q < 1, A > 0. \quad (6)$$

$$(1 - q^A) = 1 - \frac{1}{q^{-A}} = -\frac{(1 - q^{-A})}{q^{-A}} < 0, \quad \text{if } 0 < q < 1, \quad A < 0. \quad (7)$$

Note: Observe that

- i) $f(0) = 1$.
- ii) $(q^{-M}; q)_h = (1 - q^{-M})(1 - q^{-M+1}) \dots (1 - q^{-M+h-1})$, $h = 1, 2, \dots, M$, then according to (7) all the factors of this expression are negative.
- iii) $(q; q)_h = (1 - q)(1 - q^2) \dots (1 - q^h)$, $h = 1, 2, \dots, M$, then according to (6) all the factors of this expression are positive.

2 Effect of the parameters b_j , $j = 1, 2, \dots, r$ on the function ${}_{r+1}\phi_r [\cdot; q, q^n x]$

In this section the effect of the parameters b_j , $j = 1, 2, \dots, r$ on the function $f(x) \equiv {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \dots, q^{b_r+m_r} \\ q^{b_1}, \dots, q^{b_r} \end{matrix}; q, q^n x \right]$, where $0 < q < 1$, is studied.

The parameters b_j , $j = 1, 2, \dots, r$ influence in the sign and the value of the factors of

$$(q^{b_j}; q)_h = (1 - q^{b_j})(1 - q^{b_j+1}) \dots (1 - q^{b_j+h-1})$$

and

$$(q^{b_j+m_j}; q)_h = (1 - q^{b_j+m_j})(1 - q^{b_j+m_j+1}) \dots (1 - q^{b_j+m_j+h-1}),$$

where $h = 1, 2, \dots, M$.

Now we present two examples that show the effect of b_j , $j = 1, 2, \dots, r$ on the factors of $(q^{b_j}; q)_h$.

1) Let be $b_j = 3.5$, $M = 4$.

From (1) we have

$$(q^{3.5}; q)_h = (1 - q^{3.5}) \dots (1 - q^{2.5+h}), \quad h = 1, 2, 3, 4.$$

Table 1. Factors and sign of $(q^{3.5}; q)_h$ for different values of h

h	$(q^{3.5}; q)_h$	Sign of $(q^{3.5}; q)_h$
1	$(1 - q^{3.5})$	+
2	$(1 - q^{3.5})(1 - q^{4.5})$	+
3	$(1 - q^{3.5})(1 - q^{4.5})(1 - q^{5.5})$	+
4	$(1 - q^{3.5})(1 - q^{4.5})(1 - q^{5.5})(1 - q^{6.5})$	+

Therefore, according to (6), all the factors of $(q^{3.5}; q)_h$, $h = 1, 2, 3, 4$, are positive.

2) Let be $b_j = -2.3$, $M = 5$.

From (1)

$$(q^{-2.3}; q)_h = (1 - q^{-2.3})(1 - q^{-1.3}) \cdots (1 - q^{-3.3+h}), \quad h = 1, 2, 3, 4, 5.$$

Table 2. Factors and sign of $(q^{-2.3}; q)_h$ for different values of h

h	$(q^{-2.3}; q)_h$	Sign of $(q^{-2.3}; q)_h$
1	$(1 - q^{-2.3})$	-
2	$(1 - q^{-2.3})(1 - q^{-1.3})$	+
3	$(1 - q^{-2.3})(1 - q^{-1.3})(1 - q^{-0.3})$	-
4	$(1 - q^{-2.3})(1 - q^{-1.3})(1 - q^{-0.3})(1 - q^{0.7})$	-
5	$(1 - q^{-2.3})(1 - q^{-1.3})(1 - q^{-0.3})(1 - q^{0.7})(1 - q^{1.7})$	-

According to (7) and (6), all factors of $(q^{-2.3}; q)_h$, $h = 1, 2, 3$, are negative, whereas $(q^{-2.3}; q)_h$, $h = 4, 5$, have 3 negative factors and $h - 3$ positive factors.

For the general analysis we shall consider the following factors with their respective conditions, in all cases $j = 1, 2, \dots, r$; $h = 1, 2, \dots, M$.

Table 3. Conditions for determinate the possibles values of b_j

	Factor	Condition
i)	$(1 - q^{b_j+h-1})$	$b_j + h - 1 \neq 0.$
ii)	$(1 - q^{b_j+m_j+h-1})$	$b_j + m_j + h - 1 \neq 0.$

2.1 Example: Let be $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-M}, q^{b+m} \\ q^b \end{matrix} ; q, q^n x \right]$,
 $q = 0.8$, $M = 3$, $m = 5$, $n = 2$.

From (5) we have

$$f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{b+5} \\ 0.8^b \end{matrix} ; 0.8, 0.8^2 x \right] = 1 +$$

$$\frac{(1 - 0.8^{-3})(1 - 0.8^{b+5})}{(1 - 0.8)(1 - 0.8^b)} (0.8^2 x) + \frac{(1 - 0.8^{-3})(1 - 0.8^{-2})}{(1 - 0.8)(1 - 0.8^2)} \times$$

$$\frac{(1 - 0.8^{b+5})(1 - 0.8^{b+6})}{(1 - 0.8^b)(1 - 0.8^{b+1})} (0.8^2 x)^2 + \frac{(1 - 0.8^{-3})(1 - 0.8^{-2})}{(1 - 0.8)(1 - 0.8^2)} \times$$

$$\frac{(1 - 0.8^{-1})(1 - 0.8^{b+5})(1 - 0.8^{b+6})(1 - 0.8^{b+7})}{(1 - 0.8^3)(1 - 0.8^b)(1 - 0.8^{b+1})(1 - 0.8^{b+2})} (0.8^2 x)^3,$$

with the following conditions: (see Table 3)

i) $b \neq 0$, $b \neq -1$; $b \neq -2$. ii) $b \neq -5$, $b \neq -6$; $b \neq -7$.

Therefore the possible values of b are:

$b < -7$, $-7 < b < -6$, $-6 < b < -5$, $-5 < b < -2$, $-2 < b < -1$, $-1 < b < 0$, $b > 0$.

Now we present the graphic representation of $f(x)$ for different values of b .

Case 1. $b < -7$:

$$b = -9.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-4.5} \\ 0.8^{-9.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.15040x^2 -$$

$$0.71967x - 8.5037 \times 10^{-3}x^3 + 1.$$

Roots: 10.925, 4.1963, 2.5652.

$$b = -8.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-3.5} \\ 0.8^{-8.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.10992x^2 -$$

$$0.63739x - 4.3881 \times 10^{-3}x^3 + 1.$$

Roots: 17.489, 4.9024, 2.6580.

$$b = -8.0 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-3.0} \\ 0.8^{-8.0} \end{matrix} ; 0.8, 0.8^2 x \right] = 8.7477 \times 10^{-2} x^2 - 0.58604x - 2.5474 \times 10^{-3} x^3 + 1.$$

Roots: 26.102, 5.5059, 2.7315.

$$b = -7.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-2.5} \\ 0.8^{-7.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 6.4044 \times 10^{-2} x^2 - 0.52598x - 1.0276 \times 10^{-3} x^3 + 1.$$

Roots: 53.017, 6.4728, 2.8359.

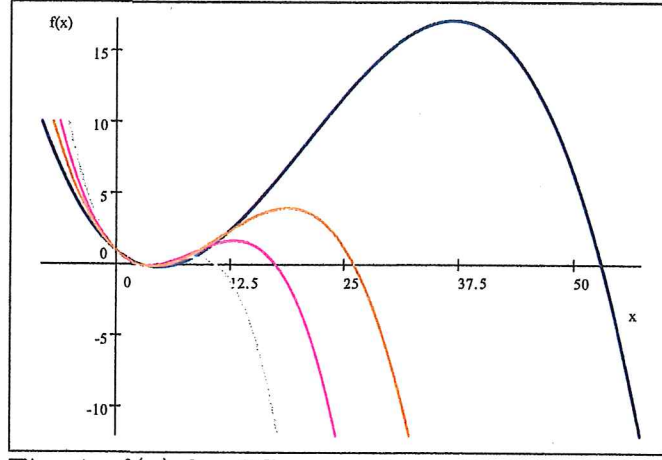


Fig. 1. $f(x)$ for different values of b with $b < -7$
gray $b = -9.5$ magenta $b = -8.5$ sienna $b = -8.0$ blue $b = -7.5$

Case 2. $-7 < b < -6$:

$$b = -6.9 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-1.9} \\ 0.8^{-6.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 3.5811 \times 10^{-2} x^2 - 0.43964x + 1.3057 \times 10^{-4} x^3 + 1.$$

Roots: 8.8299, 3.0314, -286.14.

$$b = -6.8 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-1.8} \\ 0.8^{-6.8} \end{matrix} ; 0.8, 0.8^2 x \right] = 3.1252 \times 10^{-2} x^2 - 0.42346x + 2.3311 \times 10^{-4} x^3 + 1.$$

Roots: 9.5108, 3.0757, -146.65.

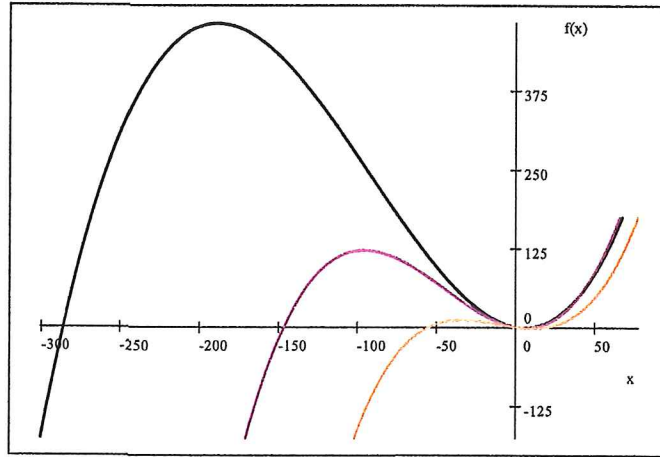


Fig. 2. $f(x)$ for different values of b with $-7 < b < -6$
black $b = -6.9$ purple $b = -6.8$ sienna $b = -6.3$

$$b = -6.3 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-1.3} \\ 0.8^{-6.3} \end{matrix} ; 0.8, 0.8^2 x \right] = 1.0200 \times 10^{-2} x^2 - 0.33339x + 3.0030 \times 10^{-4} x^3 + 1.$$

Roots: 17.829, 3.3849, -55.179.

Case 3. $-6 < b < -5$:

$$b = -5.9 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-0.9} \\ 0.8^{-5.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 1 - 2.7627 \times 10^{-3} x^2 - 1.4209 \times 10^{-4} x^3 - 0.24844x.$$

Roots: $-11.637 + 41.258i$, $-11.637 - 41.258i$, 3.8299.

$$b = -5.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-0.5} \\ 0.8^{-5.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 1 - 9.1106 \times 10^{-3} x^2 - 7.1786 \times 10^{-4} x^3 - 0.14926x.$$

Roots: 4.7800, $-8.7357 + 14.667i$, $-8.7357 - 14.667i$.

$$b = -5.1 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-0.1} \\ 0.8^{-5.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 1 - 3.9459 \times 10^{-3} x^2 - 4.4832 \times 10^{-4} x^3 - 3.2455 \times 10^{-2} x.$$

Roots: 9.2832, $-9.0424 + 12.59i$, $-9.0424 - 12.59i$.

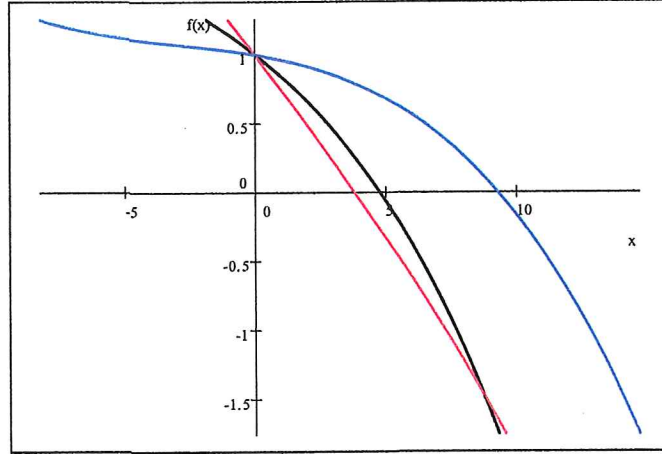


Fig. 3. $f(x)$ for different values of b with $-6 < b < -5$
red $b = -5.9$ black $b = -5.5$ blue $b = -5.1$

Case 4. $-5 < b < -2$:

$$b = -4.9 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{0.1} \\ 0.8^{-4.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 3.3917 \times 10^{-2}x + 5.3203 \times 10^{-3}x^2 + 7.1713 \times 10^{-4}x^3 + 1.$$

Roots: -12.528 , $2.5546 + 10.236i$, $2.5546 - 10.236i$.

$$b = -4.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{0.5} \\ 0.8^{-4.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.18617x + 4.4741 \times 10^{-2}x^2 + 8.3971 \times 10^{-3}x^3 + 1.$$

Roots: -5.347 , $9.4493 \times 10^{-3} + 4.7193i$, $9.4493 \times 10^{-3} - 4.7193i$.

$$b = -4.0 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{1.0} \\ 0.8^{-4.0} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.42320x + 0.15984x^2 + 4.5467 \times 10^{-2}x^3 + 1.$$

Roots: -2.9126 , $-0.30153 + 2.7314i$, $-0.30153 - 2.7314i$.

$$b = -3.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{1.5} \\ 0.8^{-3.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.73298x + 0.41958x^2 + 0.18757x^3 + 1.$$

Roots: -1.7469 , $-0.24497 - 1.7297i$, $-0.24497 + 1.7297i$.

$$b = -3.0 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{2.0} \\ 0.8^{-3.0} \end{matrix} ; 0.8, 0.8^2 x \right] = 1.152x + 0.99942x^2 + 0.77385x^3 + 1.$$

Roots: $-0.12186 + 1.1038i$, $-0.12186 - 1.1038i$, -1.0478 .

$$b = -2.1 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{2.9} \\ 0.8^{-2.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 2.431x + 5.0783x^2 + 49.062x^3 + 1.$$

Roots: $7.0043 \times 10^{-2} + 0.28066i$, $7.0043 \times 10^{-2} - 0.28066i$, -0.24359 .

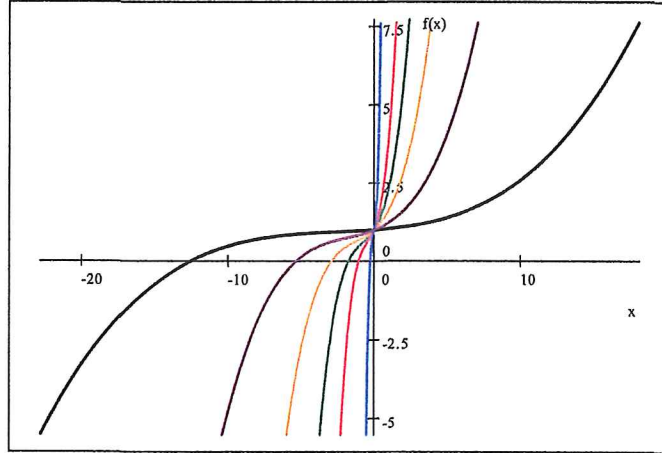


Fig. 4. $f(x)$ for different values of b with $-5 < b < -2$
 black $b = -4.9$ purple $b = -4.5$ sienna -4.0 green $b = -3.5$
 red -3.0 blue $b = -2.1$

Case 5. $-2 < b < -1$:

$$b = -1.9 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.1} \\ 0.8^{-1.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 2.8841x + 7.7730x^2 - 78.481x^3 + 1.$$

Roots: 0.32874 , $-0.11485 + 0.15991i$, $-0.11485 - 0.15991i$.

$$b = -1.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.5} \\ 0.8^{-1.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 4.1587x + 22.325x^2 - 49.013x^3 + 1.$$

Roots: 0.63845 , $-9.1476 \times 10^{-2} - 0.15359i$, $-9.1476 \times 10^{-2} + 0.15359i$.

$$b = -1.1 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.9} \\ 0.8^{-1.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 6.3713x + 187.74x^2 - 247.62x^3 + 1.$$

Roots: $-1.9325 \times 10^{-2} - 6.8517 \times 10^{-2}i$, $-1.9325 \times 10^{-2} + 6.8517 \times 10^{-2}i$, 0.79683 .

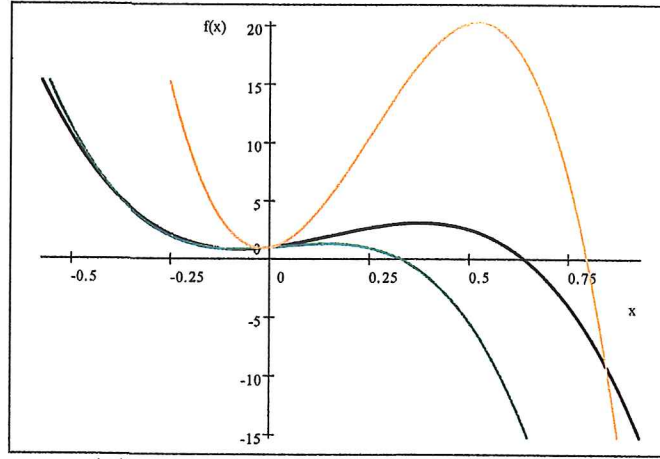


Fig. 5. $f(x)$ for different values of b with $-2 < b < -1$
green $b = -1.9$ black $b = -1.5$ sienna $b = -1.1$

Case 6. $-1 < b < 0$:

$$b = -0.9 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{4.1} \\ 0.8^{-0.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 8.2201x - 253.14x^2 + 283.56x^3 + 1.$$

Roots: 0.85391 , 8.6519×10^{-2} , -4.7734×10^{-2} .

$$b = -0.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{4.5} \\ 0.8^{-0.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 16.373x - 109.64x^2 + 96.738x^3 + 1.$$

Roots: 0.942, 0.23752, -0.0462.

$$b = -0.1 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{4.9} \\ 0.8^{-0.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 89.874x - 361.55x^2 + 269.47x^3 + 1.$$

Roots: 1.0067, 0.34562, -1.0665×10^{-2} .

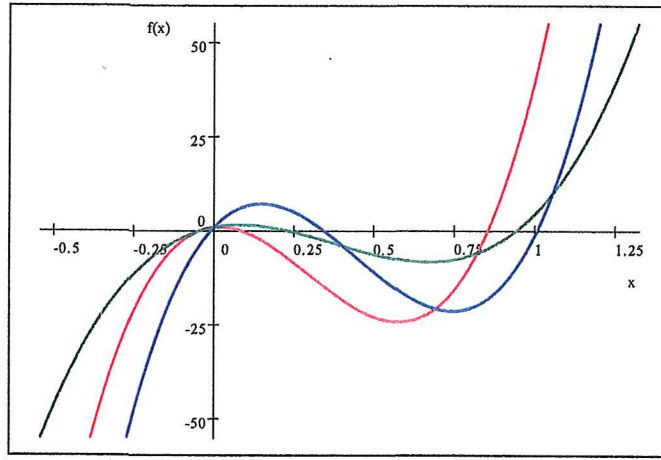


Fig. 6. $f(x)$ for different values of b with $-1 < b < 0$
red $b = -0.9$ green $b = -0.5$ blue $b = -0.1$

Case 7. $b > 0$:

$$b = 0.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{5.5} \\ 0.8^{0.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 54.962x^2 - 20.423x - 34.24x^3 + 1.$$

Roots: 1.0761, 0.47148, 5.7562×10^{-2} .

$$b = 1.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{6.5} \\ 0.8^{1.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 15.596x^2 - 8.2081x - 8.0181x^3 + 1.$$

Roots: 1.1482, 0.6225, 0.17450.

$$b = 3.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{8.5} \\ 0.8^{3.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 6.6414x^2 - 4.7824x - 2.7845x^3 + 1.$$

Roots: 1.2143, 0.80215, 0.36870.

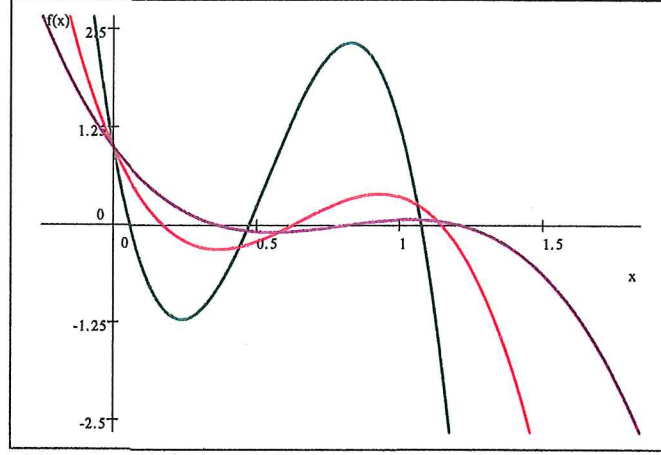


Fig. 7. $f(x)$ for different values of b with $b > 0$
green $b = 0.5$ red $b = 1.5$ purple $b = 3.5$

The previous results are compiled in the following Table.

Table 4. Characteristics of $f(x)$ for different values of b

Case	Polynomial	Roots	Set of Positivity
1 and 7	$a_2x^2 - a_1x - a_3x^3 + 1$	3 prr: r_1, r_2, r_3	$(-\infty; r_1) \cup (r_2; r_3)$.
2	$a_2x^2 - a_1x + a_3x^3 + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.
3	$1 - a_2x^2 - a_3x^3 - a_1x$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
4	$a_1x + a_2x^2 + a_3x^3 + 1$	1 nrr: r_1 2 ir	$(r_1; +\infty)$.
5	$a_1x + a_2x^2 - a_3x^3 + 1$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
6	$a_1x - a_2x^2 + a_3x^3 + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.

In this Table and through of all paper is used the following notation:

nrr: negative real root.

pr: positive real root.

ir: imaginary roots.

$a_1, a_2, a_3, \dots$ positive constants.

$r_1 < r_2 < r_3 < \dots$

2.2 Example: $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2} \\ q^{b_1}, q^{b_2} \end{matrix} ; q, q^n x \right],$
 $q = 0.6, M = 2, m_1 = 2, m_2 = 3, n = 1.$

From (5) we can write

$$f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{b_1+2}, 0.6^{b_2+3} \\ 0.6^{b_1}, 0.6^{b_2} \end{matrix} ; 0.6, 0.6x \right] = 1 + \frac{(1 - 0.6^{-2})}{(1 - 0.6)} \times$$

$$\frac{(1 - 0.6^{b_1+2})(1 - 0.6^{b_2+3})}{(1 - 0.6^{b_1})(1 - 0.6^{b_2})} (0.6x) + \frac{(1 - 0.6^{-2})(1 - 0.6^{-1})}{(1 - 0.6)(1 - 0.6^2)} \times$$

$$\frac{(1 - 0.6^{b_1+2})(1 - 0.6^{b_1+3})(1 - 0.6^{b_2+3})(1 - 0.6^{b_2+4})}{(1 - 0.6^{b_1})(1 - 0.6^{b_1+1})(1 - 0.6^{b_2})(1 - 0.6^{b_2+1})} (0.6x)^2.$$

Using Table 3 we establish the following conditions:

i) $b_1 \neq 0, -1.$ ii) $b_1 \neq -2, -3.$ i) $b_2 \neq 0, -1.$ ii) $b_2 \neq -3, -4.$

Therefore the possible values of b_1 y b_2 are:

$b_1 < -3, -3 < b_1 < -2, -2 < b_1 < -1, -1 < b_1 < 0, b_1 > 0.$

$b_2 < -4, -4 < b_2 < -3, -3 < b_2 < -1, -1 < b_2 < 0, b_2 > 0.$

We consider the following cases:

Case 1.1. $b_1 < -3, b_2 < -4:$

$$b_1 = -5.8, b_2 = -6.2 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-3.8}, 0.6^{-3.2} \\ 0.6^{-5.8}, 0.6^{-6.2} \end{matrix} ; 0.6, 0.6x \right] =$$

$$4.6221 \times 10^{-3}x^2 - 0.15738x + 1.$$

Roots: 25.597, 8.4522.

Case 1.2. $b_1 < -3, -4 < b_2 < -3 :$

$$b_1 = -8.4, b_2 = -3.3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-6.4}, 0.6^{-0.3} \\ 0.6^{-8.4}, 0.6^{-3.3} \end{matrix} ; 0.6, 0.6x \right] =$$

$$1 - 1.0219 \times 10^{-3}x^2 - 3.5271 \times 10^{-2}x.$$

Roots: 18.469, -52.985.

Case 1.3. $b_1 < -3, -3 < b_2 < -1 :$

$$b_1 = -6.1, b_2 = -2.3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-4.1}, 0.6^{0.7} \\ 0.6^{-6.1}, 0.6^{-2.3} \end{matrix} ; 0.6, 0.6x \right] =$$

$$0.11833x + 1.4067 \times 10^{-2}x^2 + 1.$$

Roots: $-4.206 + 7.3074i$, $-4.206 - 7.3074i$.

$$b_1 = -3.2, b_2 = -1.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-1.2}, 0.6^{1.6} \\ 0.6^{-3.2}, 0.6^{-1.4} \end{matrix} ; 0.6, 0.6x \right] =$$

$$0.29217x + 3.0669 \times 10^{-2}x^2 + 1.$$

Roots: $-4.7633 + 3.1492i$, $-4.7633 - 3.1492i$.

Case 1.4. $b_1 < -3, -1 < b_2 < 0 :$

$$b_1 = -7.6, b_2 = -0.5 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-5.6}, 0.6^{2.5} \\ 0.6^{-7.6}, 0.6^{-0.5} \end{matrix} ; 0.6, 0.6x \right] =$$

$$2.2901x - 1.7832x^2 + 1.$$

Roots: 1.6286, -0.34434.

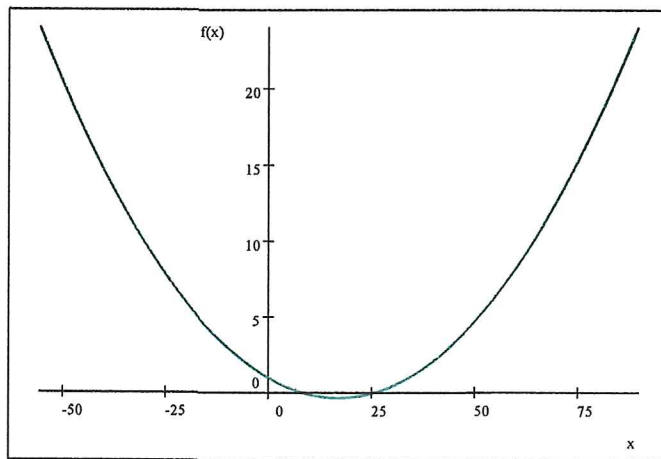
Case 1.5. $b_1 < -3, b_2 > 0 :$

$$b_1 = -4.7, b_2 = 3.9 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-2.7}, 0.6^{6.9} \\ 0.6^{-4.7}, 0.6^{3.9} \end{matrix} ; 0.6, 0.6x \right] =$$

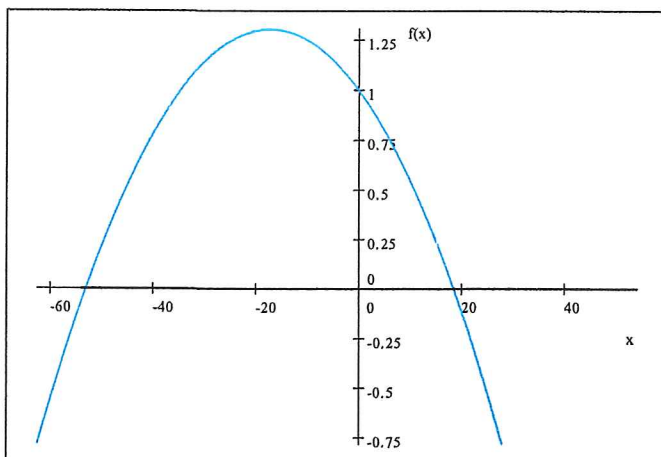
$$0.14609x^2 - 0.88770x + 1.$$

Roots: 4.5827, 1.4937.

Fig. 8. Case 1.1-Case 1.5: $f(x)$ for different values of b_1, b_2

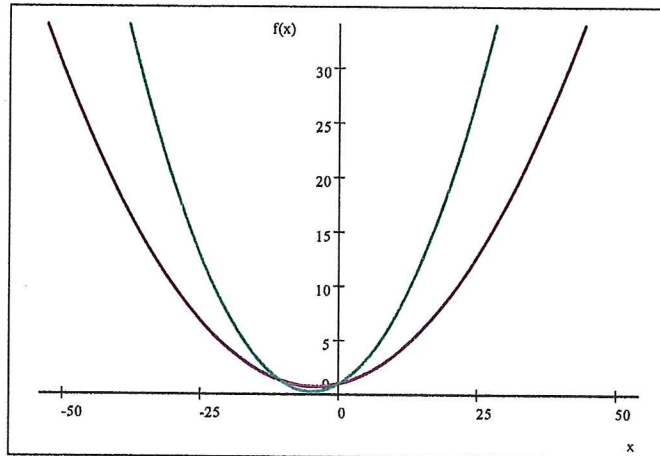


$$b_1 = -5.8, b_2 = -6.2$$

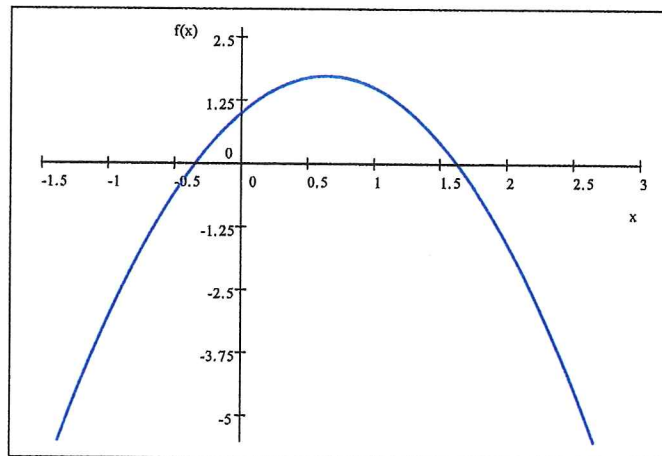


$$b_1 = -8.4, b_2 = -3.3$$

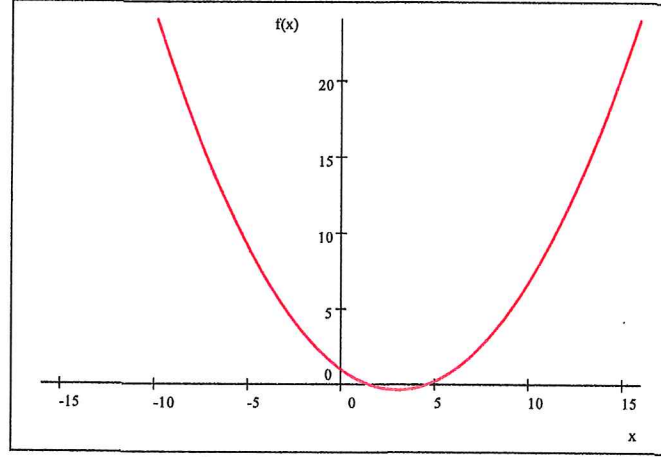
- 44 -



purple $b_1 = -6.1$, $b_2 = -2.3$ green $b_1 = -3.2$, $b_2 = -1.4$



$b_1 = -7.6$, $b_2 = -0.5$



$$b_1 = -4.7, b_2 = 3.9$$

Case 2.1. $-3 < b_1 < -2, b_2 < -4$:

$$b_1 = -2.8, b_2 = -4.2 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.8}, 0.6^{-1.2} \\ 0.6^{-2.8}, 0.6^{-4.2} \end{matrix} ; 0.6, 0.6x \right] =$$

$$1 - 4.9779 \times 10^{-5}x^2 - 4.7455 \times 10^{-2}x.$$

Roots: 20.626, -973.95.

Case 2.2. $-3 < b_1 < -2, -4 < b_2 < -3$:

$$b_1 = -2.5, b_2 = -3.2 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.5}, 0.6^{-0.2} \\ 0.6^{-2.5}, 0.6^{-3.2} \end{matrix} ; 0.6, 0.6x \right] =$$

$$1.5452 \times 10^{-4}x^2 - 7.8196 \times 10^{-3}x + 1.$$

Roots: $25.302 + 76.363i, 25.302 - 76.363i$.

Case 2.3. $-3 < b_1 < -2, -3 < b_2 < -1$:

$$b_1 = -2.1, b_2 = -1.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.1}, 0.6^{1.4} \\ 0.6^{-2.1}, 0.6^{-1.6} \end{matrix} ; 0.6, 0.6x \right] =$$

$$2.9360 \times 10^{-2}x - 1.7669 \times 10^{-2}x^2 + 1.$$

Roots: 8.3997, -6.738.

$$b_1 = -2.4, b_2 = -2.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.4}, 0.6^{0.4} \\ 0.6^{-2.4}, 0.6^{-2.6} \end{matrix} ; 0.6, 0.6x \right] =$$
$$1.6729 \times 10^{-2}x - 1.0677 \times 10^{-3}x^2 + 1.$$

Roots: 39.426, -23.757.

Case 2.4. $-3 < b_1 < -2, -1 < b_2 < 0 :$

$$b_1 = -2.9, b_2 = -0.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.9}, 0.6^{2.6} \\ 0.6^{-2.9}, 0.6^{-0.4} \end{matrix} ; 0.6, 0.6x \right] =$$
$$1.4846x + 8.9799 \times 10^{-2}x^2 + 1.$$

Roots: -0.70350, -15.829.

$$b_1 = -2.5, b_2 = -0.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.5}, 0.6^{2.6} \\ 0.6^{-2.5}, 0.6^{-0.4} \end{matrix} ; 0.6, 0.6x \right] =$$
$$0.97288x + 0.37915x^2 + 1.$$

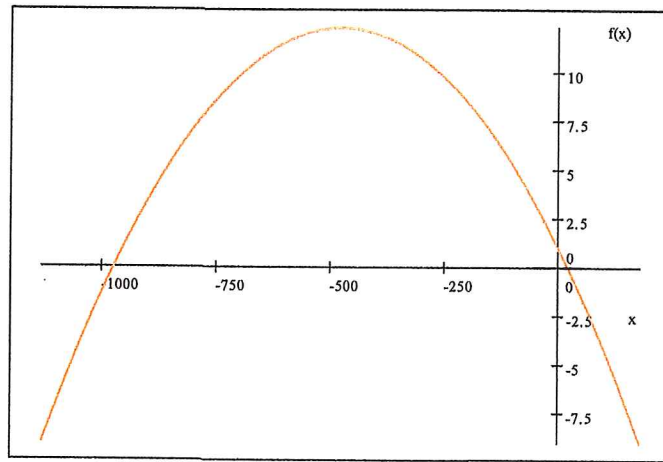
Roots: $-1.2830 + 0.99573i$, $-1.2830 - 0.99573i$.

Case 2.5. $-3 < b_1 < -2, b_2 > 0 :$

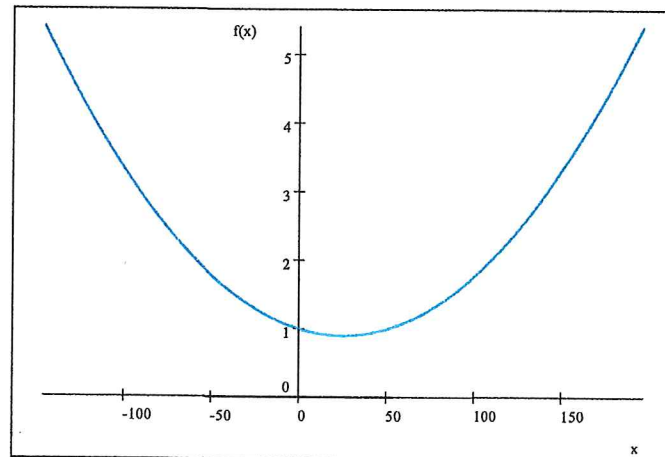
$$b_1 = -2.3, b_2 = 5.7 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{-0.3}, 0.6^{8.7} \\ 0.6^{-2.3}, 0.6^{5.7} \end{matrix} ; 0.6, 0.6x \right] =$$
$$1 - 4.2196 \times 10^{-2}x^2 - 0.20625x.$$

Roots: 3.0032, -7.8911.

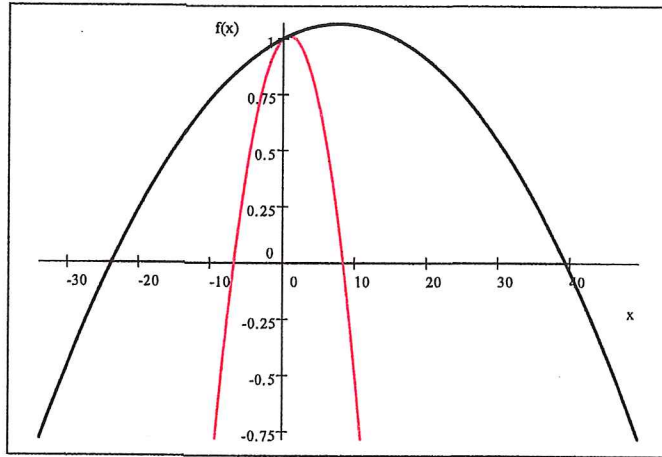
Fig. 9. Case 2.1-Case 2.5: $f(x)$ for different values of b_1, b_2



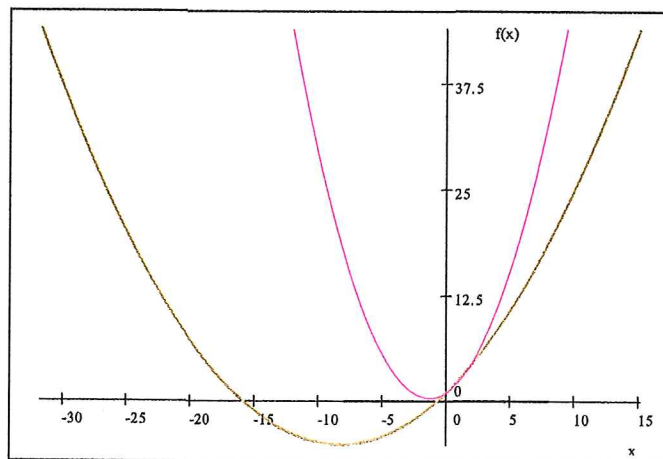
$$b_1 = -2.8, b_2 = -4.2$$



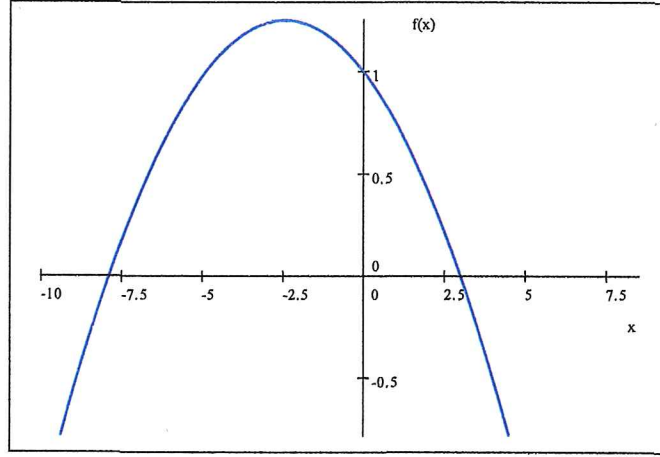
$$b_1 = -2.5, b_2 = -3.2$$



red $b_1 = -2.1$, $b_2 = -1.6$ black $b_1 = -2.4$, $b_2 = -2.6$



green $b_1 = -2.9$, $b_2 = -0.4$ magenta $b_1 = -2.5$, $b_2 = -0.4$



$$b_1 = -2.3, b_2 = 5.7$$

Case 3.1. $-2 < b_1 < -1, b_2 < -4$:

$$b_1 = -1.1, b_2 = -7.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{0.9}, 0.6^{-4.6} \\ 0.6^{-1.1}, 0.6^{-7.6} \end{matrix} ; 0.6, 0.6x \right] =$$

$$0.26004x + 0.36235x^2 + 1.$$

Roots: $-0.35883 + 1.622i, -0.35883 - 1.622i$.

Case 3.2. $-2 < b_1 < -1, -4 < b_2 < -3$:

$$b_1 = -1.5, b_2 = -3.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{0.5}, 0.6^{-0.6} \\ 0.6^{-1.5}, 0.6^{-3.6} \end{matrix} ; 0.6, 0.6x \right] =$$

$$3.5385 \times 10^{-2}x - 0.00271x^2 + 1.$$

Roots: $26.817, -13.760$.

Case 3.3. $-2 < b_1 < -1, -3 < b_2 < -1$:

$$b_1 = -1.9, b_2 = -2.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{0.1}, 0.6^{0.6} \\ 0.6^{-1.9}, 0.6^{-2.4} \end{matrix} ; 0.6, 0.6x \right] =$$

$$2.1857 \times 10^{-3}x^2 - 8.8818 \times 10^{-3}x + 1.$$

Roots: $2.0318 + 21.293i, 2.0318 - 21.293i$.

$$b_1 = -1.7, b_2 = -1.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{0.3}, 0.6^{1.6} \\ 0.6^{-1.7}, 0.6^{-1.4} \end{matrix} ; 0.6, 0.6x \right] =$$

$$0.335x^2 - 0.14645x + 1.$$

Roots: $0.21858 + 1.7139i$, $0.21858 - 1.7139i$.

Case 3.4. $-2 < b_1 < -1, -1 < b_2 < 0 :$

$$b_1 = -1.4, b_2 = -0.2 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{0.6}, 0.6^{2.8} \\ 0.6^{-1.4}, 0.6^{-0.2} \end{matrix} ; 0.6, 0.6x \right] =$$

$$1 - 18.734x^2 - 4.7665x.$$

Roots: 0.13653 , -0.39097 .

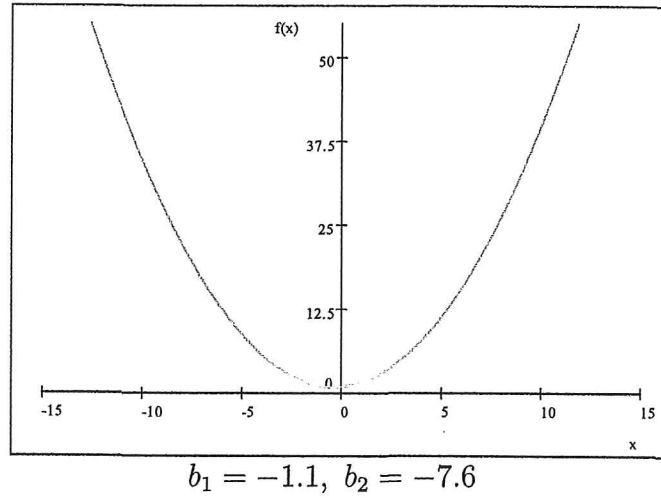
Case 3.5. $-2 < b_1 < -1, b_2 > 0 :$

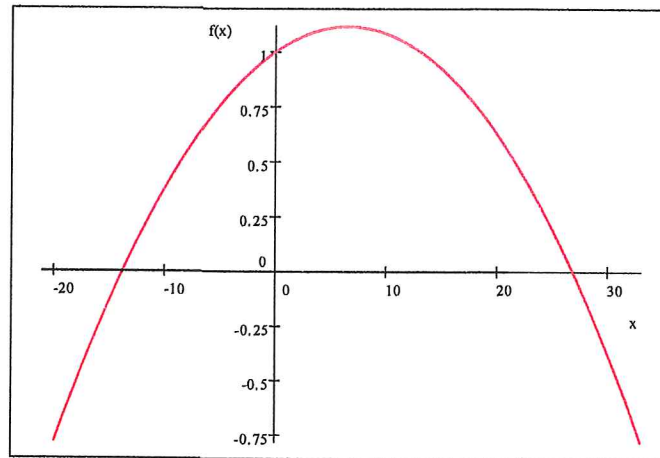
$$b_1 = -1.8, b_2 = 4.5 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{0.2}, 0.6^{7.5} \\ 0.6^{-1.8}, 0.6^{4.5} \end{matrix} ; 0.6, 0.6x \right] =$$

$$0.18677x + 0.11129x^2 + 1.$$

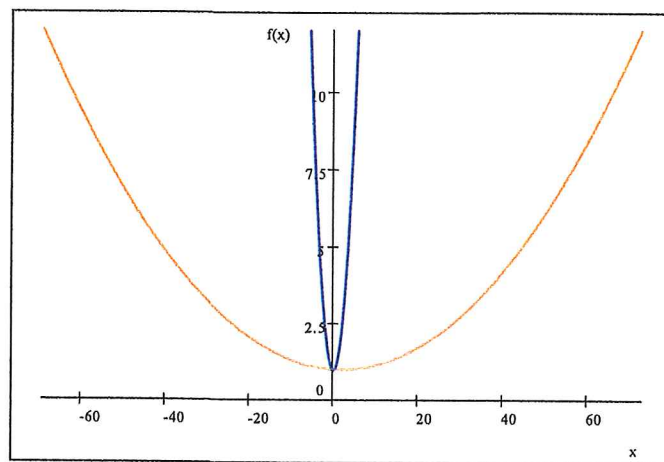
Roots: $-0.83906 + 2.8777i$, $-0.83906 - 2.8777i$.

Fig. 10. Case 3.1-Case 3.5: $f(x)$ for different values of b_1, b_2

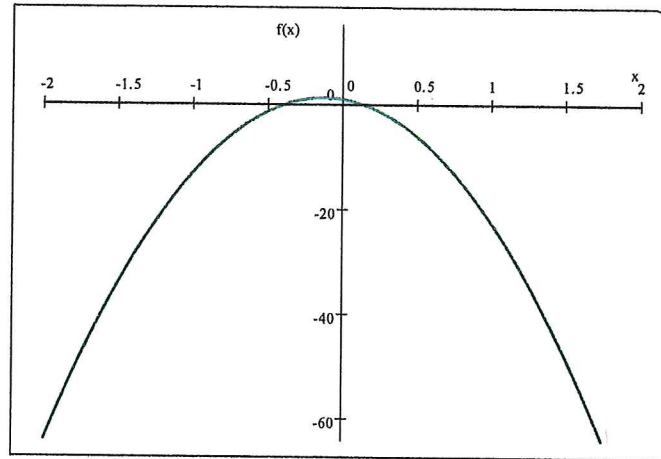




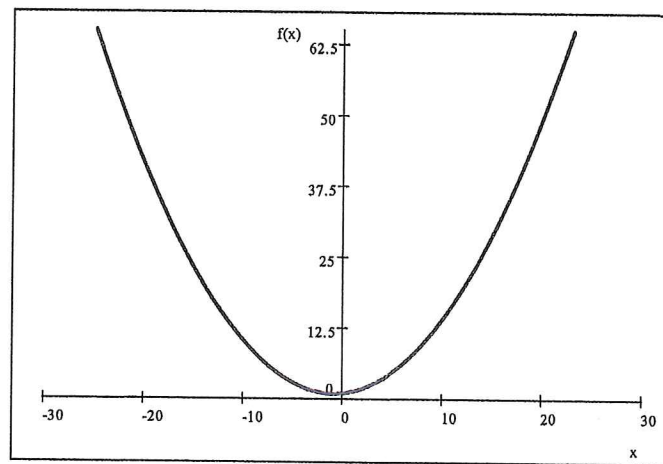
$$b_1 = -1.5, b_2 = -3.6$$



sienna $b_1 = -1.9, b_2 = -2.4$ blue $b_1 = -1.7, b_2 = -1.4$



$$b_1 = -1.4, b_2 = -0.2$$



$$b_1 = -1.8, b_2 = 4.5$$

Case 4.1. $-1 < b_1 < 0, b_2 < -4$:

$$b_1 = -0.5, b_2 = -4.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.5}, 0.6^{-1.6} \\ 0.6^{-0.5}, 0.6^{-4.6} \end{matrix} ; 0.6, 0.6x \right] =$$

$$0.65397x - 8.8658 \times 10^{-2}x^2 + 1.$$

Roots: 8.6764, -1.3.

Case 4.2. $-1 < b_1 < 0, -4 < b_2 < -3 :$

$$b_1 = -0.8, b_2 = -3.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.2}, 0.6^{-0.4} \\ 0.6^{-0.8}, 0.6^{-3.4} \end{matrix} ; 0.6, 0.6x \right] =$$
$$0.11729x + 5.5862 \times 10^{-2}x^2 + 1.$$

Roots: $-1.0498 + 4.0987i, -1.0498 - 4.0987i$.

Case 4.3. $-1 < b_1 < 0, -3 < b_2 < -1 :$

$$b_1 = -0.4, b_2 = -2.0 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.6}, 0.6^{1.0} \\ 0.6^{-0.4}, 0.6^{-2.0} \end{matrix} ; 0.6, 0.6x \right] =$$
$$1 - 2.4690x^2 - 1.4778x.$$

Roots: $0.40399, -1.0026$.

$$b_1 = -0.9, b_2 = -1.3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.1}, 0.6^{1.7} \\ 0.6^{-0.9}, 0.6^{-1.3} \end{matrix} ; 0.6, 0.6x \right] =$$
$$1 - 45.109x^2 - 1.2092x.$$

Roots: $0.13609, -0.16290$.

Case 4.4. $-1 < b_1 < 0, -1 < b_2 < 0 :$

$$b_1 = -0.1, b_2 = -0.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.9}, 0.6^{2.4} \\ 0.6^{-0.1}, 0.6^{-0.6} \end{matrix} ; 0.6, 0.6x \right] =$$
$$363.70x^2 - 62.258x + 1.$$

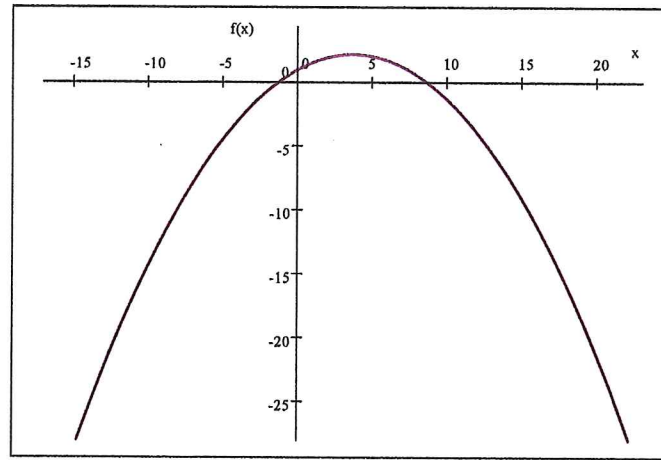
Roots: $0.15324, 1.7943 \times 10^{-2}$.

Case 4.5. $-1 < b_1 < 0, b_2 > 0 :$

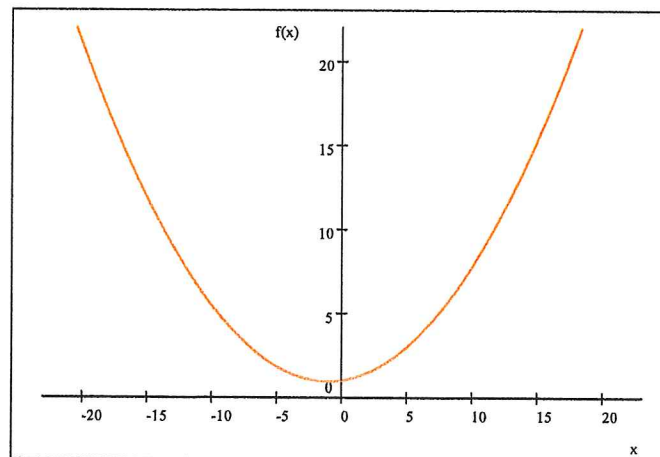
$$b_1 = -0.3, b_2 = 6.7 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.7}, 0.6^{9.7} \\ 0.6^{-0.3}, 0.6^{6.7} \end{matrix} ; 0.6, 0.6x \right] =$$
$$9.5922x - 15.155x^2 + 1.$$

Roots: $0.72409, -0.09113$.

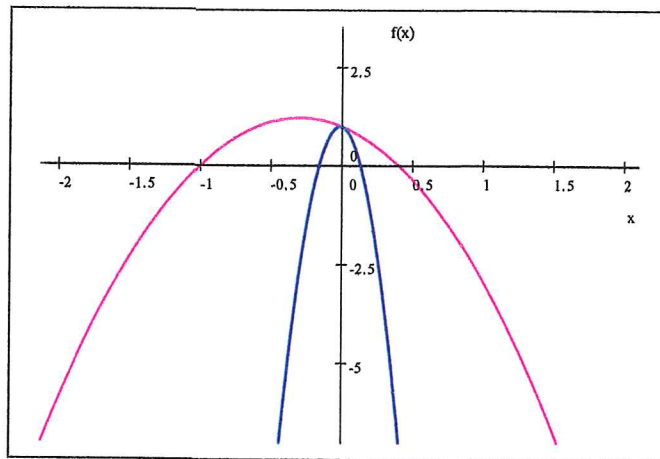
Fig. 11. Case 4.1-Case 4.5: $f(x)$ for different values of b_1, b_2



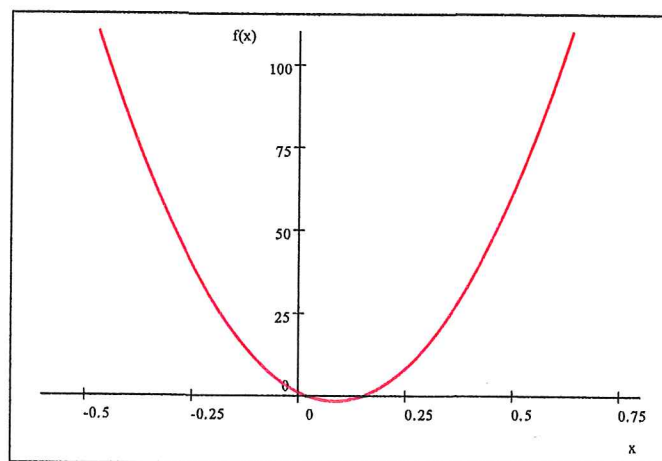
$$b_1 = -0.5, b_2 = -4.6$$



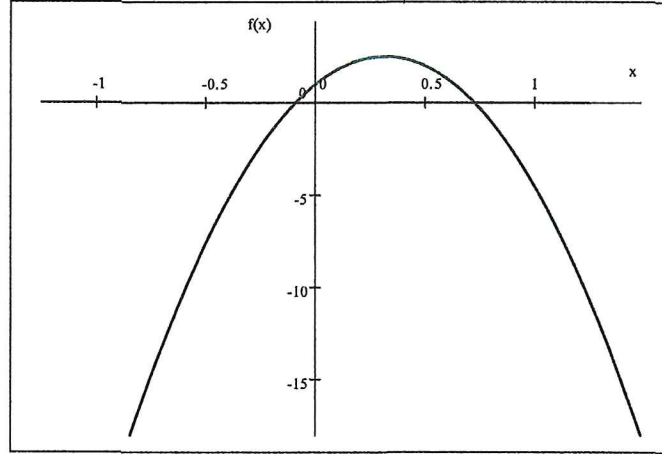
$$b_1 = -0.8, b_2 = -3.4$$



magenta $b_1 = -0.4$, $b_2 = -2.0$ blue $b_1 = -0.9$, $b_2 = -1.3$



$b_1 = -0.1$, $b_2 = -0.6$



$$b_1 = -0.3, b_2 = 6.7$$

Case 5.1. $b_1 > 0, b_2 < -4$:

$$b_1 = 3.5, b_2 = -5.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{5.5}, 0.6^{-2.4} \\ 0.6^{3.5}, 0.6^{-5.4} \end{matrix} ; 0.6, 0.6x \right] =$$

$$4.0517 \times 10^{-2}x^2 - 0.49038x + 1.$$

Roots: 9.5069, 2.5961.

Case 5.2. $b_1 > 0, -4 < b_2 < -3$:

$$b_1 = 6.9, b_2 = -3.8 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{8.9}, 0.6^{-0.8} \\ 0.6^{6.9}, 0.6^{-3.8} \end{matrix} ; 0.6, 0.6x \right] =$$

$$1 - 4.4406 \times 10^{-3}x^2 - 0.22999x.$$

Roots: 4.0338, -55.826.

Case 5.3. $b_1 > 0, -3 < b_2 < -1$:

$$b_1 = 5.2, b_2 = -2.9 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{7.2}, 0.6^{0.1} \\ 0.6^{5.2}, 0.6^{-2.9} \end{matrix} ; 0.6, 0.6x \right] =$$

$$4.0957 \times 10^{-2}x + 6.9011 \times 10^{-3}x^2 + 1.$$

Roots: $-2.9675 + 11.666i, -2.9675 - 11.666i$.

$$b_1 = 4.1, b_2 = -1.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{6.1}, 0.6^{1.4} \\ 0.6^{4.1}, 0.6^{-1.6} \end{matrix} ; 0.6, 0.6x \right] =$$

$$1.1743x + 1.5196x^2 + 1.$$

Roots: $-0.38638 + 0.71328i, -0.38638 - 0.71328i$.

Case 5.4. $b_1 > 0, -1 < b_2 < 0$:

$$b_1 = 1.2, b_2 = -0.7 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{3.2}, 0.6^{2.3} \\ 0.6^{1.2}, 0.6^{-0.7} \end{matrix} ; 0.6, 0.6x \right] =$$

$$7.5313x - 35.308x^2 + 1.$$

Roots: $0.30589, -9.2588 \times 10^{-2}$.

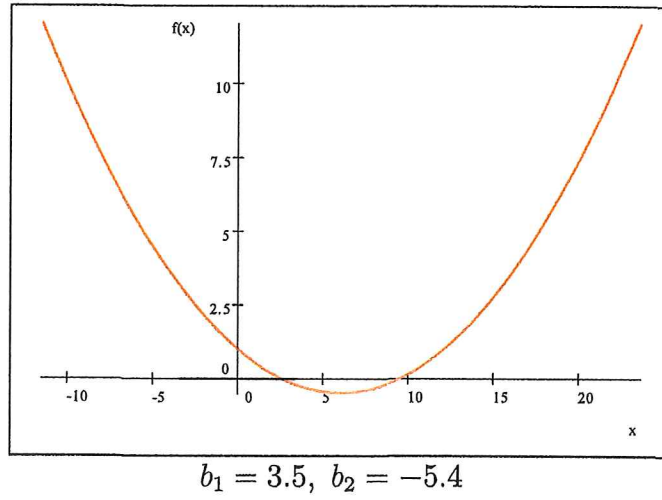
Case 5.5. $b_1 > 0, b_2 > 0$:

$$b_1 = 2.7, b_2 = 6.3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{4.7}, 0.6^{9.3} \\ 0.6^{2.7}, 0.6^{6.3} \end{matrix} ; 0.6, 0.6x \right] =$$

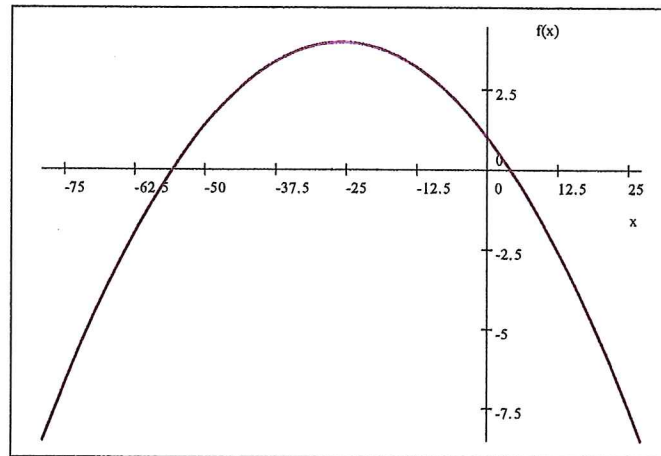
$$2.3750x^2 - 3.3469x + 1.$$

Roots: $0.97925, 0.42998$.

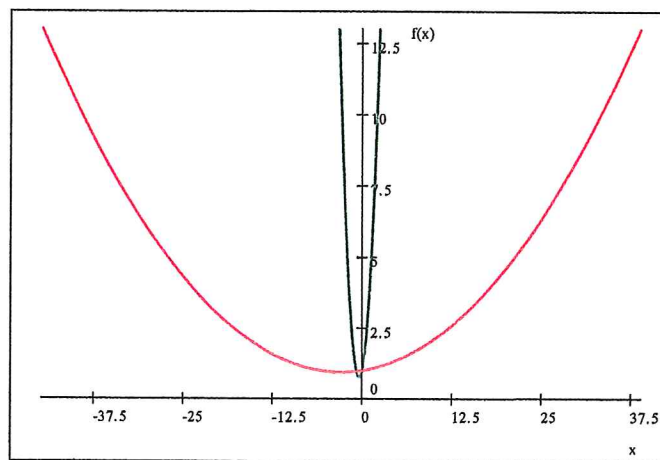
Fig. 12. Case 5.1-Case 5.5: $f(x)$ for different values of b_1, b_2



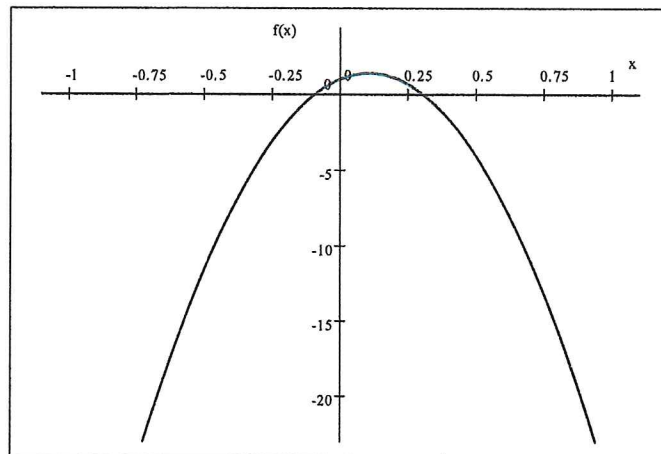
- 58 -



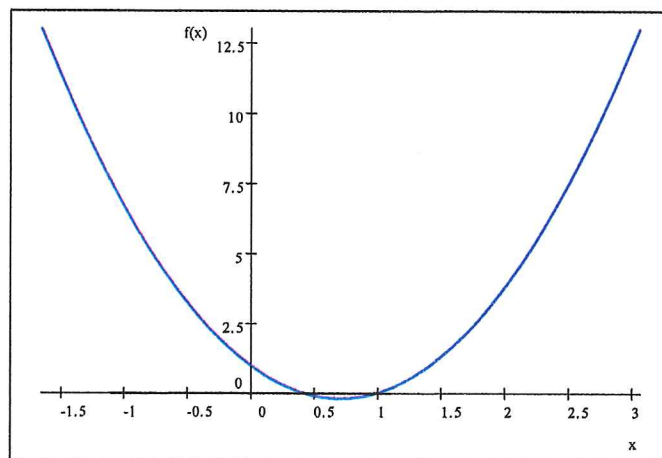
$$b_1 = 6.9, b_2 = -3.8$$



red $b_1 = 5.2, b_2 = -2.9$ green $b_1 = 4.1, b_2 = -1.6$



$$b_1 = 1.2, b_2 = -0.7$$



$$b_1 = 2.7, b_2 = 6.3$$

The previous results are compiled in the following Table.

Table 5. Characteristics of $f(x)$ for different values of b_1, b_2

Case	Polynomial	Roots	Set of Positivity
1.1 and 5.1 1.5 and 5.5	$a_2x^2 - a_1x + 1$	2 prr: r_1, r_2	$(-\infty; r_1) \cup (r_2; +\infty)$.
1.2 and 5.2	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
1.3 and 5.3	$a_1x + a_2x^2 + 1$	2 ir	$(-\infty; +\infty)$.
1.4 and 5.4	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
2.1 and 2.5	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
2.2	$a_2x^2 - a_1x + 1$	2 ir	$(-\infty; +\infty)$.
2.3	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
2.4	$a_1x + a_2x^2 + 1$	2 nrr: r_1, r_2 2 ir	$(-\infty; r_1) \cup (r_2; +\infty)$. $(-\infty; +\infty)$.
3.1 and 3.5	$a_1x + a_2x^2 + 1$	2 ir	$(-\infty; +\infty)$.
3.2	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
3.3	$a_2x^2 - a_1x + 1$	2 ir	$(-\infty; +\infty)$.
3.4	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
4.1 and 4.5	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
4.2	$a_1x + a_2x^2 + 1$	2 ir	$(-\infty; +\infty)$.
4.3	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
4.4	$a_2x^2 - a_1x + 1$	2 prr: r_1, r_2	$(-\infty; r_1) \cup (r_2; +\infty)$.

3 Effect of the parameters m_j , $j = 1, 2, \dots, r$ on the function ${}_{r+1}\phi_r [\cdot ; q, q^n x]$

The parameters m_j , $j = 1, 2, \dots, r$ influence in the sign and the value of the factors of $(q^{b_j+m_j}; q)_h$, $h = 1, 2, \dots, M$.

Observe that if $b_j > 0$ then $b_j + m_j > 0$ and according to (6) all the factors of $(q^{b_j+m_j}; q)_h$ are positive.

Now we give two examples to show the effect of m_j , $j = 1, 2, \dots, r$ on the factors of $(q^{b_j+m_j}; q)_h$.

1) Let be $b_j = -6.2, m_j = 5, M = 4$.

From (1) for $b_j + m_j = -1.2$ and $M = 4$ we obtain

$$(q^{-1.2}; q)_h = (1 - q^{-1.2}) (1 - q^{-0.2}) \dots (1 - q^{-2.2+h}), \quad h = 1, 2, 3, 4.$$

Table 6. Factors and sign of $(q^{-1.2}; q)_h$ for different values of h

h	$(q^{-1.2}; q)_h$	Sign of $(q^{-1.2}; q)_h$
1	$(1 - q^{-1.2})$	-
2	$(1 - q^{-1.2}) (1 - q^{-0.2})$	+
3	$(1 - q^{-1.2}) (1 - q^{-0.2}) (1 - q^{0.8})$	+
4	$(1 - q^{-1.2}) (1 - q^{-0.2}) (1 - q^{0.8}) (1 - q^{1.8})$	+

Therefore, according to (7) and (6), all factors of $(q^{-1.2}; q)_h$, $h = 1, 2$, are negative, whereas $(q^{-1.2}; q)_h$, $h = 3, 4$, have 2 negative factors and $h - 2$ positive factors.

2) Let be $b_j = -6.2, m_j = 2, M = 4$.

From (1) for $b_j + m_j = -4.2$ and $M = 4$ we have

$$(q^{-4.2}; q)_h = (1 - q^{-4.2}) (1 - q^{-3.2}) \dots (1 - q^{-5.2+h}), \quad h = 1, 2, 3, 4.$$

Table 7. Factors and sign of $(q^{-4.2}; q)_h$ for different values of h

h	$(q^{-4.2}; q)_h$	Sign of $(q^{-4.2}; q)_h$
1	$(1 - q^{-4.2})$	-
2	$(1 - q^{-4.2})(1 - q^{-3.2})$	+
3	$(1 - q^{-4.2})(1 - q^{-3.2})(1 - q^{-2.2})$	-
4	$(1 - q^{-4.2})(1 - q^{-3.2})(1 - q^{-2.2})(1 - q^{-1.2})$	+

In this case all factors of $(q^{-4.2}; q)_h$, $h = 1, 2, 3, 4$, are negative, according to (7).

3.1 Example: Let be $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-M}, q^{b+m} \\ q^b \end{matrix}; q, q^n x \right]$,
 $q = 0.5$, $M = 3$, $n = 1$.

From (5) we have

$$\begin{aligned}
 f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{b+m} \\ 0.5^b \end{matrix}; 0.5, 0.5x \right] &= 1 + \frac{(1 - 0.5^{-3})(1 - 0.5^{b+m})}{(1 - 0.5)(1 - 0.5^b)} \times \\
 (0.5x) &+ \frac{(1 - 0.5^{-3})(1 - 0.5^{-2})(1 - 0.5^{b+m})(1 - 0.5^{b+m+1})}{(1 - 0.5)(1 - 0.5^2)(1 - 0.5^b)(1 - 0.5^{b+1})} (0.5x)^2 + \\
 &\frac{(1 - 0.5^{-3})(1 - 0.5^{-2})(1 - 0.5^{-1})}{(1 - 0.5)(1 - 0.5^2)(1 - 0.5^3)} \times \\
 &\frac{(1 - 0.5^{b+m})(1 - 0.5^{b+m+1})(1 - 0.5^{b+m+2})}{(1 - 0.5^b)(1 - 0.5^{b+1})(1 - 0.5^{b+2})} (0.5x)^3. \quad (8)
 \end{aligned}$$

3.1.1 Let be $b = 0.1$.

From (8) for different values of m we get,

$$\begin{aligned}
 m = 1 : f(x) &= {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{1.1} \\ 0.5^{0.1} \end{matrix}; 0.5, 0.5x \right] = 160.29x^2 - 55.765x - \\
 &105.53x^3 + 1.
 \end{aligned}$$

Roots: 0.9999, 0.50005, 1.8952×10^{-2} .

$$m = 2 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{2.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 265.42x^2 - 80.147x - 186.28x^3 + 1.$$

Roots: 0.99994, 0.41187, 1.3035×10^{-2} .

$$m = 3 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{3.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 325.98x^2 - 92.338x - 235.86x^3 + 1.$$

Roots: 0.99159, 0.37923, 1.1275×10^{-2} .

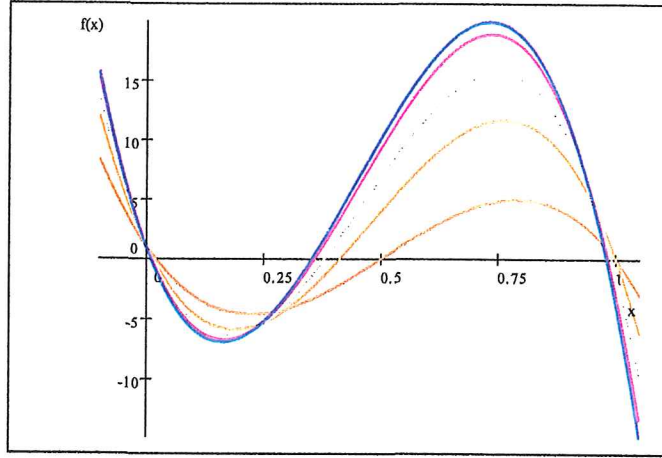


Fig. 13. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{0.1+m} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right]$ for different values of m
 sienna $m = 1$ orange $m = 2$ gray $m = 3$ magenta $m = 5$ blue
 $m = 10$

$$m = 5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{5.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 374.9x^2 - 101.48x - 277.37x^3 + 1.$$

Roots: 0.98326, 0.35813, 1.0238×10^{-2} .

$$m = 10 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{10.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 391.34x^2 - 104.43x - 291.59x^3 + 1.$$

Roots: 0.98033, 0.35181, 9.9436×10^{-3} .

Analogously we obtain the following results:

3.1.2 Let be $b = -1.7$

$$m = 1 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-0.7} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 2.1125x^3 - 1.1687x^2 - 1.9438x + 1.$$

Roots: 1.0, 0.5, -0.94675 .

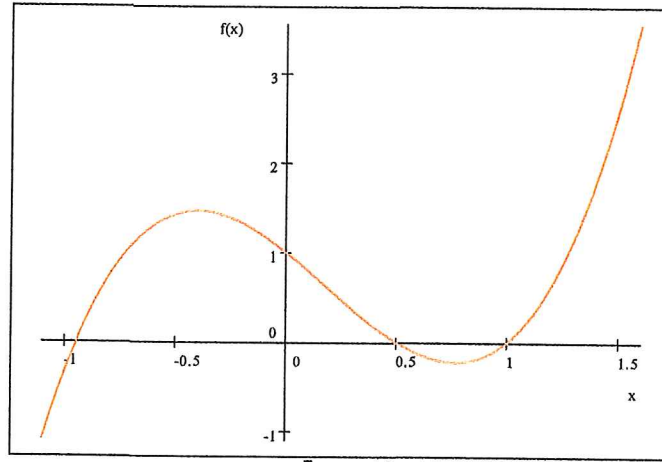


Fig. 14. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-0.7} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right]$

$$m = 2 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{0.3} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 0.58436x + 1.1114x^2 - 2.6958x^3 + 1.$$

Roots: 1.0, $-0.29386 + 0.53348i$, $-0.29386 - 0.53348i$.

$$m = 3 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{1.3} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 1.8484x + 4.7176x^2 - 12.901x^3 + 1.$$

Roots: $0.71659, -0.17545 + 0.27819i, -0.17545 - 0.27819i$.

$$m = 4 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{2.3} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 2.4805x + 7.1372x^2 - 20.620x^3 + 1.$$

Roots: $0.64755, -0.15071 - 0.22843i, -0.15071 + 0.22843i$.

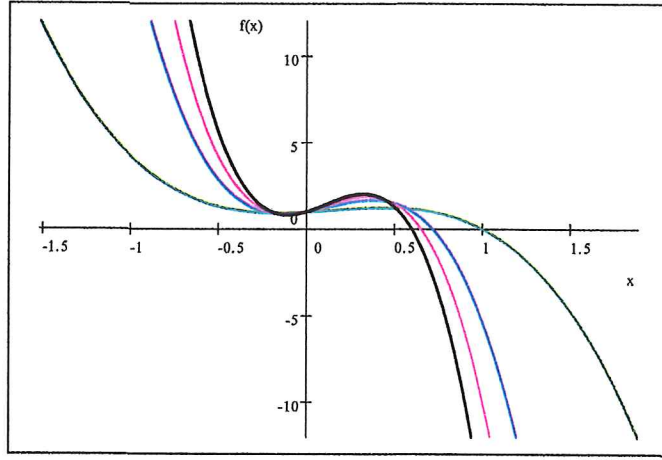


Fig. 15. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-1.7+m} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right]$ for different values of m

green $m = 2$ blue $m = 3$ magenta $m = 4$ black $m = 8$

$$m = 8 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{6.3} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 3.0730x + 9.7789x^2 - 29.669x^3 + 1.$$

Roots: $0.59742, -0.13391 - 0.19618i, -0.13391 + 0.19618i$.

3.1.3 Let be $b = -4.5$:

$$m = 1 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-3.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 3.0145x^2 - 3.3382x - 0.67634x^3 + 1.$$

Roots: $2.9571, 1.0, 0.5$.

$$m = 2 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-2.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 0.53442x^2 - 1.5073x - 2.7163 \times 10^{-2}x^3 + 1.$$

Roots: 16.435, 2.2401, 1.0.

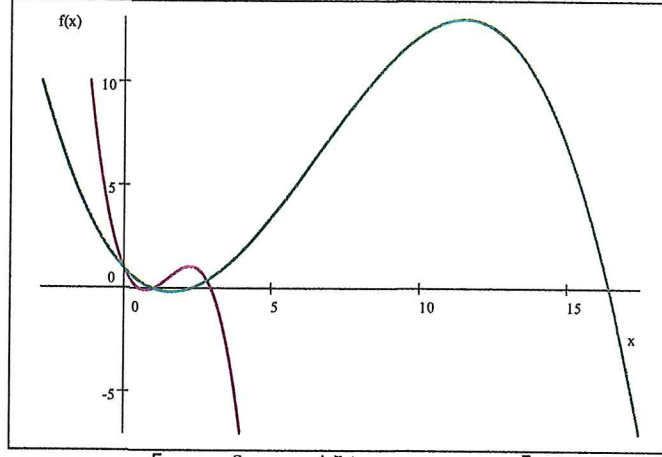


Fig. 16. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-4.5+m} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right]$ for different values
of m
purple $m = 1$ green $m = 2$

$$m = 3 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-1.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 4.7535 \times 10^{-2}x^2 - 0.59179x + 1.7084 \times 10^{-3}x^3 + 1.$$

Roots: 7.6049, 2.0535, -37.482.

$$m = 4 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-0.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 1 - 7.6145 \times 10^{-3}x^2 - 6.0401 \times 10^{-4}x^3 - 0.13407x.$$

Roots: 5.2457, $-8.9262 + 15.36i$, $-8.9262 - 15.36i$.

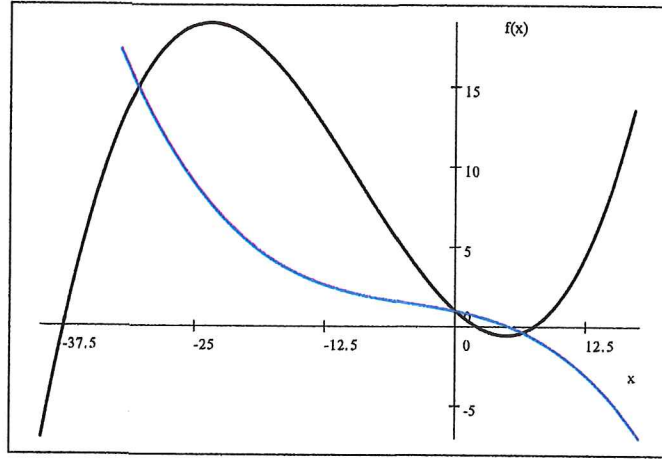


Fig. 17. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-4.5+m} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right]$ for different values
of m
black $m = 3$ blue $m = 4$

$$m = 5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{0.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 9.4799 \times 10^{-2}x + 1.1884 \times 10^{-2}x^2 + 1.2004 \times 10^{-3}x^3 + 1.$$

Roots: $-10.18, 0.14037 - 9.0448i, 0.14037 + 9.0448i$.

$$m = 7 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{2.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 0.26645x + 4.7102 \times 10^{-2}x^2 + 5.5243 \times 10^{-3}x^3 + 1.$$

Roots: $-5.6569, -1.4347 + 5.4719i, -1.4347 - 5.4719i$.

$$m = 12 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{7.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 0.32188x + 6.2245 \times 10^{-2}x^2 + 7.6273 \times 10^{-3}x^3 + 1.$$

Roots: $-4.9753, -1.5927 - 4.8801i, -1.5927 + 4.8801i$.

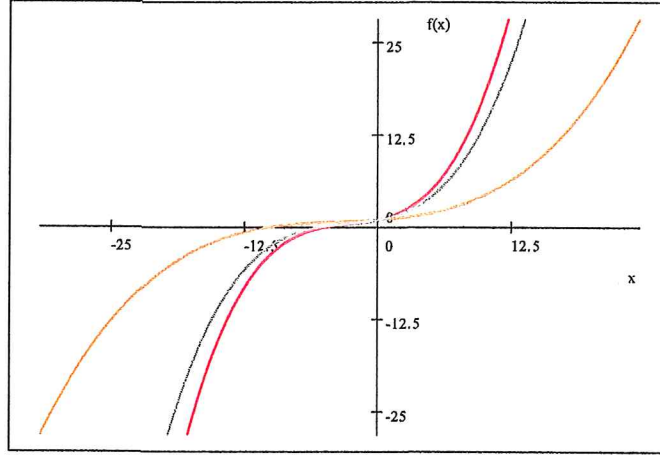


Fig. 18. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-4.5+m} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right]$ for different values
of m
sienna $m = 5$ gray $m = 7$ red $m = 12$

Table 8. Characteristics of $f(x)$ for different values of m with b fixed.

b	m	Polynomial	Roots	Set of Positivity
0.1	$1, 2, 3, \dots$	$a_2x^2 - a_1x - a_3x^3 + 1$	3 prr: r_1, r_2, r_3	$(-\infty; r_1) \cup (r_2; r_3)$.
-1.7	1	$a_3x^3 - a_2x^2 - a_1x + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.
	$2, 3, 4, \dots$	$a_1x + a_2x^2 - a_3x^3 + 1$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
-4.5	1, 2	$a_2x^2 - a_1x - a_3x^3 + 1$	3 prr: r_1, r_2, r_3	$(-\infty; r_1) \cup (r_2; r_3)$.
	3	$a_2x^2 - a_1x + a_3x^3 + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.
	4	$1 - a_2x^2 - a_3x^3 - a_1x$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
	$5, 6, 7, \dots$	$a_1x + a_2x^2 + a_3x^3 + 1$	1 nrr: r_1 2 ir	$(r_1; +\infty)$.

3.2 Example: $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2} \\ q^{b_1}, q^{b_2} \end{matrix} ; q, q^n x \right],$
 $q = 0.3, M = 4, b_1 = -2.9, b_2 = -6.4, m_2 = 3, n = 4.$

From (5) we get

$$\begin{aligned} f(x) &= {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-2.9+m_1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = \\ &1 + \frac{(1 - 0.3^{-4})(1 - 0.3^{-2.9+m_1})(1 - 0.3^{-3.4})}{(1 - 0.3)(1 - 0.3^{-2.9})(1 - 0.3^{-6.4})} (0.3^4 x) + \\ &\frac{(1 - 0.3^{-4})(1 - 0.3^{-3})(1 - 0.3^{-2.9+m_1})(1 - 0.3^{-1.9+m_1})}{(1 - 0.3)(1 - 0.3^2)(1 - 0.3^{-2.9})(1 - 0.3^{-1.9})} \times \\ &\frac{(1 - 0.3^{-3.4})(1 - 0.3^{-2.4})}{(1 - 0.3^{-6.4})(1 - 0.3^{-5.4})} (0.3^4 x)^2 + \\ &\frac{(1 - 0.3^{-4})(1 - 0.3^{-3})(1 - 0.3^{-2})}{(1 - 0.3)(1 - 0.3^2)(1 - 0.3^3)} \times \\ &\frac{(1 - 0.3^{-2.9+m_1})(1 - 0.3^{-1.9+m_1})(1 - 0.3^{-0.9+m_1})}{(1 - 0.3^{-2.9})(1 - 0.3^{-1.9})(1 - 0.3^{-0.9})} \times \\ &\frac{(1 - 0.3^{-3.4})(1 - 0.3^{-2.4})(1 - 0.3^{-1.4})}{(1 - 0.3^{-6.4})(1 - 0.3^{-5.4})(1 - 0.3^{-4.4})} (0.3^4 x)^3 + \\ &\frac{(1 - 0.3^{-4})(1 - 0.3^{-3})(1 - 0.3^{-2})(1 - 0.3^{-1})}{(1 - 0.3)(1 - 0.3^2)(1 - 0.3^3)(1 - 0.3^4)} \times \\ &\frac{(1 - 0.3^{-2.9+m_1})(1 - 0.3^{-1.9+m_1})(1 - 0.3^{-0.9+m_1})(1 - 0.3^{0.1+m_1})}{(1 - 0.3^{-2.9})(1 - 0.3^{-1.9})(1 - 0.3^{-0.9})(1 - 0.3^{0.1})} \times \\ &\frac{(1 - 0.3^{-3.4})(1 - 0.3^{-2.4})(1 - 0.3^{-1.4})(1 - 0.3^{-0.4})}{(1 - 0.3^{-6.4})(1 - 0.3^{-5.4})(1 - 0.3^{-4.4})(1 - 0.3^{-3.4})} (0.3^4 x)^4. \quad (9) \end{aligned}$$

Then from (9) we have:

$$m_1 = 1 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-1.9}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 1.8935 \times 10^{-5} x^2 - 1.0464 \times 10^{-2} x + 2.0441 \times 10^{-9} x^3 - 2.645 \times 10^{-12} x^4 + 1.$$

Roots: 2826.9, 417.47, 123.45, -2595.0.

$$m_1 = 2 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-0.9}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 2.75 \times 10^{-13} x^4 - 2.4267 \times 10^{-7} x^2 - 1.6952 \times 10^{-10} x^3 - 2.3116 \times 10^{-3} x + 1.$$

Roots: 412.99, 2295.5, -1046.0 - 1655.8i, -1046.0 + 1655.8i.

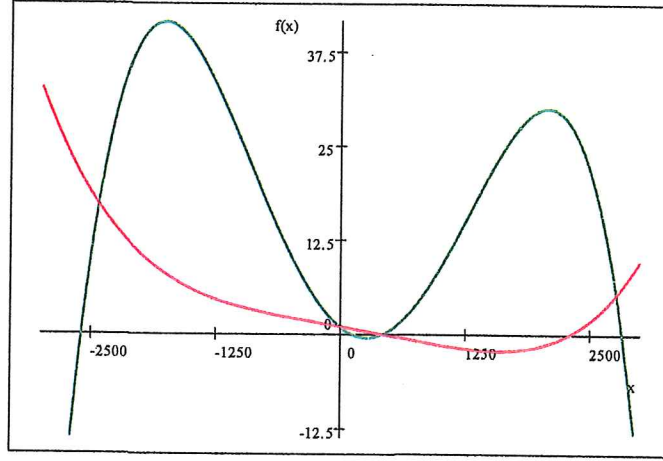


Fig. 19. $f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-2.9+m_1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right]$ for different values of m_1
green $m_1 = 1$ red $m_1 = 2$

$$m_1 = 3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{0.1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 1.341 \times 10^{-4} x + 9.1104 \times 10^{-8} x^2 + 7.9784 \times 10^{-11} x^3 - 1.3728 \times 10^{-13} x^4 + 1.$$

Roots: 36.853 + 1539.4i, 36.853 - 1539.4i, 2024.8, -1517.3.

$$m_1 = 4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{1.1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 8.678 \times 10^{-4} x + 7.3908 \times 10^{-7} x^2 + 6.8653 \times 10^{-10} x^3 - 1.2016 \times 10^{-12} x^4 + 1.$$

Roots: $-780.68, 1522.6, -85.288 + 832.38i, -85.288 - 832.38i$.

$m_1 = 5 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{2.1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 1.0879 \times 10^{-3}x + 9.8278 \times 10^{-7}x^2 + 9.2858 \times 10^{-10}x^3 - 1.6334 \times 10^{-12}x^4 + 1$.
Roots: $1474.2, -702.8, -101.45 + 761.98i, -101.45 - 761.98i$.

$m_1 = 10 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{7.1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 1.182 \times 10^{-3}x + 1.0939 \times 10^{-6}x^2 + 1.041 \times 10^{-9}x^3 - 1.8352 \times 10^{-12}x^4 + 1$.
Roots: $1456.5, -674.83, -107.2 - 736.82i, -107.2 + 736.82i$.

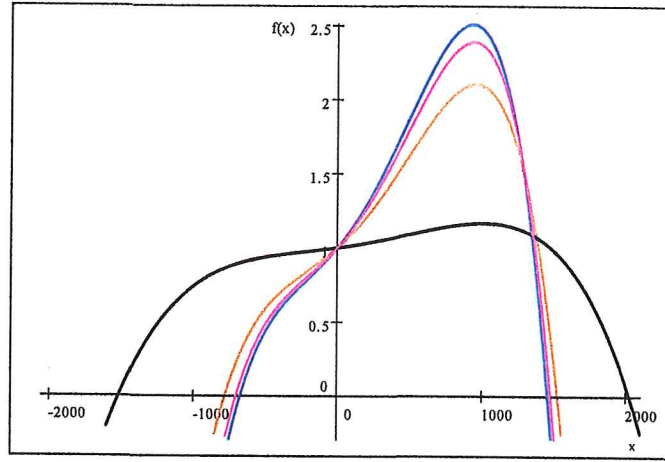


Fig. 20. $f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-2.9+m_1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right]$ for different values of m_1

black $m_1 = 3$ sienna $m_1 = 4$ magenta $m_1 = 5$ blue $m_1 = 10$

Table 9. Characteristics of $f(x)$ for different values of m_1

m_1	Polynomial	Roots	Set of Positivity
1	$a_2x^2 - a_1x + a_3x^3 - a_4x^4 + 1$	1 nrr: r_1 3 prr: r_2, r_3, r_4	$(r_1; r_2) \cup (r_3; r_4)$.
2	$a_4x^4 - a_2x^2 - a_3x^3 - a_1x + 1$	2 prr: r_1, r_2 2 ir	$(-\infty; r_1) \cup (r_2; +\infty)$.
3, 4, 5, ...	$a_1x + a_2x^2 + a_3x^3 - a_4x^4 + 1$	1 nrr: r_1 1 prr: r_2 , 2 ir	$(r_1; r_2)$.

4 Effect of q on $f(x)$

4.1 Example: $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-M}, q^{b+m} \\ q^b \end{matrix}; q, q^n x \right], M = 3, b = -2.8, m = 6, n = 3.$

From (5) we have

$$q = 0.9 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.9^{-3}, 0.9^{3.2} \\ 0.9^{-2.8} \end{matrix}; 0.9, 0.9^3 x \right] = 2.2603x + 3.4834x^2 + 4.994x^3 + 1.$$

Roots: $-0.54306, -7.7227 \times 10^{-2} - 0.60230i, -7.7227 \times 10^{-2} + 0.60230i$.

$$q = 0.8 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.2} \\ 0.8^{-2.8} \end{matrix}; 0.8, 0.8^3 x \right] = 1.4348x + 1.4125x^2 + 1.3016x^3 + 1.$$

Roots: $-0.85087, -0.11717 + 0.94298i, -0.11717 - 0.94298i$.

$$q = 0.7 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.7^{-3}, 0.7^{3.2} \\ 0.7^{-2.8} \end{matrix}; 0.7, 0.7^3 x \right] = 0.86926x + 0.52476x^2 + 0.29992x^3 + 1.$$

Roots: $-1.3901, -0.17977 - 1.5382i, -0.17977 + 1.5382i$.

$$q = 0.6 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.6^{-3}, 0.6^{3.2} \\ 0.6^{-2.8} \end{matrix}; 0.6, 0.6^3 x \right] = 0.49615x + 0.17431x^2 +$$

$$5.8970 \times 10^{-2}x^3 + 1.$$

$$\text{Roots: } -2.3971, -0.27938 + 2.645i, -0.27938 - 2.645i.$$

$$q = 0.5 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{3.2} \\ 0.5^{-2.8} \end{matrix} ; 0.5, 0.5^3x \right] = 0.26148x + 4.9805 \times 10^{-2}x^2 + 9.3394 \times 10^{-3}x^3 + 1.$$

$$\text{Roots: } -4.4491, -0.44182 - 4.8858i, -0.44182 + 4.8858i.$$

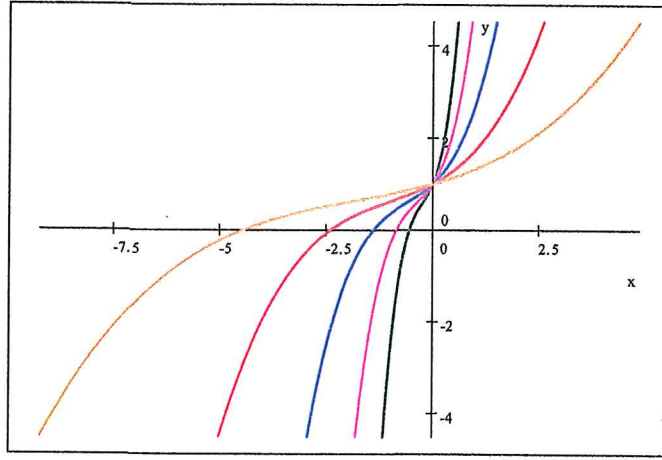


Fig. 21. $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-3}, q^{3.2} \\ q^{-2.8} \end{matrix} ; q, q^3x \right]$ for different values of q
green $q = 0.9$ magenta $q = 0.8$ blue $q = 0.7$
red $q = 0.6$ sienna $q = 0.5$

$$q = 0.4 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.4^{-3}, 0.4^{3.2} \\ 0.4^{-2.8} \end{matrix} ; 0.4, 0.4^3x \right] = 0.12298x + 1.1454 \times 10^{-2}x^2 + 1.0771 \times 10^{-3}x^3 + 1.$$

$$\text{Roots: } -9.1964, -0.71887 + 10.022i, -0.71887 - 10.022i.$$

$$q = 0.3 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.3^{-3}, 0.3^{3.2} \\ 0.3^{-2.8} \end{matrix} ; 0.3, 0.3^3x \right] = 4.8397 \times 10^{-2}x + 1.8655 \times 10^{-3}x^2 + 7.4418 \times 10^{-5}x^3 + 1.$$

$$\text{Roots: } -22.6, -1.2337 + 24.353i, -1.2337 - 24.353i.$$

$q = 0.2 : f(x) = {}_2\phi_1 \left[\begin{matrix} 0.2^{-3}, 0.2^{3.2} \\ 0.2^{-2.8} \end{matrix} ; 0.2, 0.2^3 x \right] = 1.3759 \times 10^{-2}x + 1.6056 \times 10^{-4}x^2 + 1.9735 \times 10^{-6}x^3 + 1.$
 Roots: $-76.648, -2.3548 + 81.273i, -2.3548 - 81.273i.$

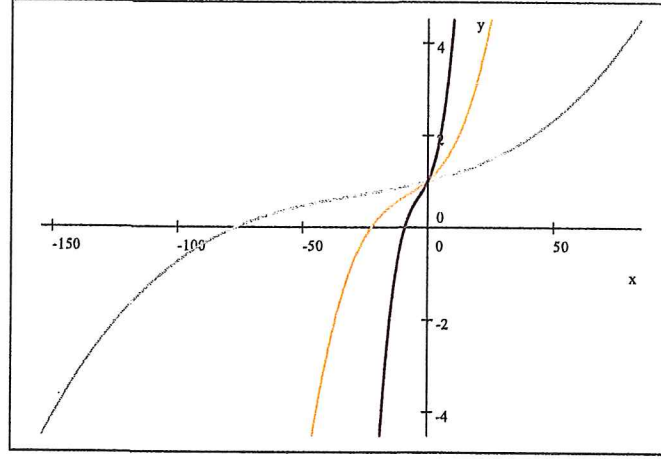


Fig. 22. $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-3}, q^{3.2} \\ q^{-2.8} \end{matrix} ; q, q^3 x \right]$ for different values of q
 purple $q = 0.4$ yellow $q = 0.3$ gray $q = 0.2$

Table 10. Characteristics of $f(x)$ for different values of q

Polynomial: $a_1x + a_2x^2 + a_3x^3 + 1.$
Roots: 1 nrr: r_1 , 2 ir.
Set of Positivity: $(r_1; +\infty).$

From Fig. 21 and Fig. 22 we observe that if q decreases, the real root of $f(x)$ moves further away from the origin.

4.2 Example: $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2} \\ q^{b_1}, q^{b_2} \end{matrix} ; q, q^n x \right],$
 $M = 4, b_1 = 0.3, m_1 = 1, b_2 = 3.5, m_2 = 2, n = 1.$

From (5) we obtain

$$q = 0.8 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.8^{-4}, 0.8^{1.3}, 0.8^{5.5} \\ 0.8^{0.3}, 0.8^{3.5} \end{matrix} ; 0.8, 0.8x \right] = 119.29x^2 - 29.24x - 164.12x^3 + 73.068x^4 + 1.$$

Roots: 1.0002, 0.76295, 0.44242, 4.0536×10^{-2} .

$$q = 0.7 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.7^{-4}, 0.7^{1.3}, 0.7^{5.5} \\ 0.7^{0.3}, 0.7^{3.5} \end{matrix} ; 0.7, 0.7x \right] = 145.62x^2 - 32.544x - 216.24x^3 + 102.17x^4 + 1.$$

Roots: 0.99968, 0.68944, 0.39104, 3.6316×10^{-2} .

$$q = 0.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-4}, 0.6^{1.3}, 0.6^{5.5} \\ 0.6^{0.3}, 0.6^{3.5} \end{matrix} ; 0.6, 0.6x \right] = 201.64x^2 - 38.83x - 336.63x^3 + 172.83x^4 + 1.$$

Roots: 0.99978, 0.59811, 0.31959, 3.0277×10^{-2} .

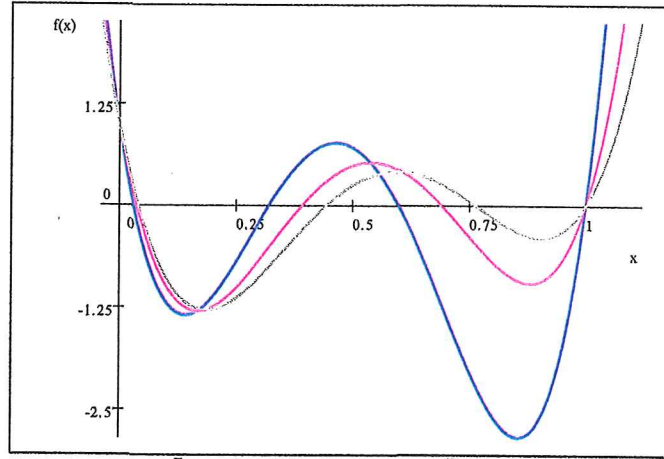


Fig. 23. $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-4}, q^{1.3}, q^{5.5} \\ q^{0.3}, q^{3.5} \end{matrix} ; q, qx \right]$ for different values of q
gray $q = 0.8$ magenta $q = 0.7$ blue $q = 0.6$

$$q = 0.5 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.5^{-4}, 0.5^{1.3}, 0.5^{5.5} \\ 0.5^{0.3}, 0.5^{3.5} \end{matrix} ; 0.5, 0.5x \right] = 329.79x^2 - 50.898x - 648.19x^3 + 368.29x^4 + 1.$$

Roots: 1.0001, 0.49969, 0.23736, 2.2892×10^{-2} .

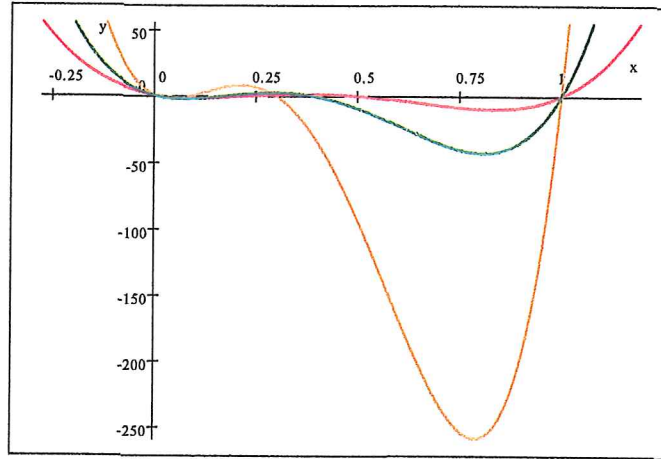


Fig.24. $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-4}, q^{1.3}, q^{5.5} \\ q^{0.3}, q^{3.5} \end{matrix} ; q, qx \right]$ for different values of q
 red $q = 0.5$ green $q = 0.4$ sienna $q = 0.3$

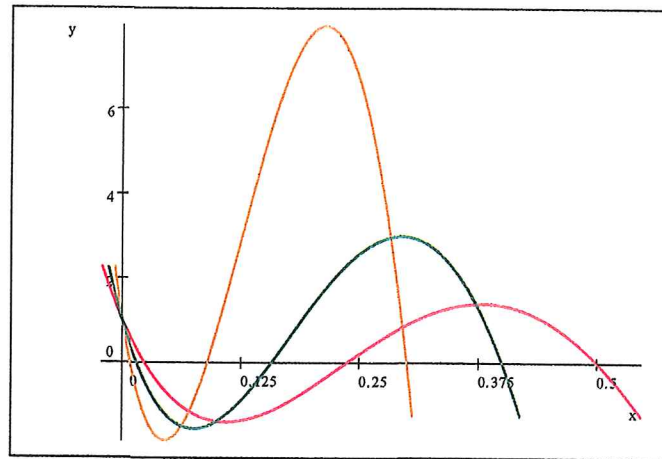


Fig.25. $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-4}, q^{1.3}, q^{5.5} \\ q^{0.3}, q^{3.5} \end{matrix} ; q, qx \right]$ for different values of q with
 $-0.0215 < x < 0.546$

$$q = 0.4 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.4^{-4}, 0.4^{1.3}, 0.4^{5.5} \\ 0.4^{0.3}, 0.4^{3.5} \end{matrix} ; 0.4, 0.4x \right] = 678.04x^2 -$$

$$76.102x - 1656.6x^3 + 1053.6x^4 + 1.$$

$$\text{Roots: } 1.0001, 0.39992, 0.15719, 1.5097 \times 10^{-2}.$$

$$q = 0.3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{1.3}, 0.3^{5.5} \\ 0.3^{0.3}, 0.3^{3.5} \end{matrix} ; 0.3, 0.3x \right] = 1962.0x^2 - 138.80x - 6410.7x^3 + 4586.5x^4 + 1.$$

$$\text{Roots: } 1.0, 0.30001, 8.9614 \times 10^{-2}, 8.1098 \times 10^{-3}.$$

Table 11. Characteristics of $f(x)$ for different values of q

Polynomial: $a_2x^2 - a_1x - a_3x^3 + a_4x^4 + 1.$
Roots: 4 prr: $r_1, r_2, r_3, r_4.$
Set of Positivity: $(-\infty; r_1) \cup (r_2; r_3) \cup (r_4; +\infty).$

From Fig. 23-Fig. 25 we observe that:

- i) if q decreases then the roots r_1, r_2, r_3 decrease, while the value of r_4 presents small downs and ups, with $r_1 < r_2 < r_3 < r_4$.
- ii) $(f_{\max})_{q=0.3} > (f_{\max})_{q=0.4} > \dots > (f_{\max})_{q=0.7} > (f_{\max})_{q=0.8}.$
 $(f_{\min})_{q=0.3} < (f_{\min})_{q=0.4} < \dots < (f_{\min})_{q=0.7} < (f_{\min})_{q=0.8}.$

5 Effect of n on the function ${}_{r+1}\phi_r [\cdot ; q, q^n x]$

5.1 Example: $f(x) = {}_4\phi_3 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, q^{b_3+m_3} \\ q^{b_1}, q^{b_2}, q^{b_3} \end{matrix} ; q, q^n x \right],$
 $q = 0.9, M = 3, b_1 = -0.5, m_1 = 3, b_2 = 2.9, m_2 = 1, b_3 = -3.4, m_3 = 2.$

Observe that q^n is a factor of scale for x and $0 < q^n < 1$.

We consider the function $g(x)$ defined by

$$g(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, x \right] = 7.5144x - 9.983x^2 - 4.6776x^3 + 1.$$

$$\text{Roots: } r_3 = 0.68182, r_2 = -0.11613, r_1 = -2.6999.$$

Then we can obtain $f(x)$ expanding $g(x)$ in $(0.9)^n$ units, since $(0.9)^n < 1$, that is,

$$f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, (0.9)^n x \right] = 7.5144((0.9)^n x) - 9.983((0.9)^n x)^2 - 4.6776((0.9)^n x)^3 + 1.$$

So the roots of $f(x)$ are $\frac{r_3}{(0.9)^n}, \frac{r_2}{(0.9)^n}, \frac{r_1}{(0.9)^n}$.

In this way we get the following results:

$$n = 1 : f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9x \right] = 6.7630x - 8.0862x^2 - 3.4100x^3 + 1.$$

Roots: 0.75758, -0.12904, -2.9999.

$$n = 2 : f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^2 x \right] = 6.0867x - 6.5499x^2 - 2.4859x^3 + 1.$$

Roots: 0.84175, -0.14338, -3.3332.

$$n = 3 : f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^3 x \right] = 5.478x - 5.3054x^2 - 1.8122x^3 + 1.$$

Roots: 0.93527, -0.15931, -3.7036.

$$n = 4 : f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^4 x \right] = 4.9302x - 4.2974x^2 - 1.3211x^3 + 1.$$

Roots: 1.0392, -0.17701, -4.1151.

$$n = 5 : f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^5 x \right] = 4.4372x - 3.4809x^2 - 0.96308x^3 + 1.$$

Roots: 1.1547, -0.19667, -4.5723.

$$n = 8 : f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^8 x \right] = 3.2347x -$$

$$1.8499x^2 - 0.37312x^3 + 1.$$

Roots: 1.5839, -0.26979, -6.272.

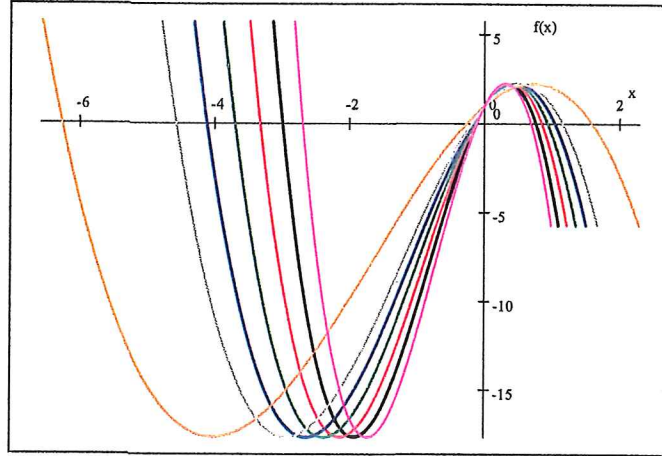


Fig. 26. $f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^n x \right]$ for different values of n

black $n = 1$ red $n = 2$ green $n = 3$ blue $n = 4$ gray $n = 5$ sienna $n = 8$
 $g(x)$: magenta

Table12. Characteristics of $f(x)$ for different values of n

Polynomial: $a_1x - a_2x^2 - a_3x^3 + 1.$
Roots: 2 nrr: r_1, r_2 ; 1 prr: $r_3.$
Set of Positivity: $(-\infty; r_1) \cup (r_2; r_3).$

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**THE CAUSAL SPIRALS ON THE SPHERES
OF LORENTZ-MINKOWSKI 3-SPACE $\mathbb{R}_1^3$**

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Abstract

Spirals are differentiable curves cutting all meridians (or all parallels) of a rotational surface with a constant angle. In this paper, we obtain differential equations of all spirals on the spheres of Lorentz-Minkowski 3-space $\mathbb{R}_1^3$ and define the general parametrizations of these curves which are the solutions of the differential equations.

keywords: Hyperbolic spherical spiral; Lorentzian spherical spirals; lightlike conical spirals; Minkowski 3-Space.

1 Introduction

The research area of the spherical curves, which has characteristic properties on sphere, attracts many authors' attention in differential geometry [4, 6, 10, 18, 20]. For instance, the curves, that are the intersection of the sphere with planes passing through the origin, give the property of shortest path on the sphere and are called geodesics. In this sense, spherical helices are important curves that make a constant angle with a fixed axis [6]. On the other hand, the spherical curves that cut the meridians (or parallels) with a constant angle are called spherical spirals (also known as loxodromes or rhumb lines), which are not geodesic [6, 11, 18, 20]. These special curves are widely used in aviation and navigation. It is well known that the spherical spirals look like straight lines and logarithmic spirals under the mercator projections [18] and the stereographic projections [6], respectively.

Spherical curves has been also studied in Lorentz-Minkowski 3-space $\mathbb{R}_1^3$ by the researchers. For example, Petrović and Šućurović [16] presented some characterizations of the Lorentzian spherical timelike and null curves. They showed that there are no non-geodesic null curves on $S_1^2(r)$. Inoguchi and Lee [8] gave some characterizations of null helices in terms of their associated curves. They examined the geometry of null curves and presented the important applications of null curves. Besides, they proved the existence of two geodesic null lines on $S_1^2(r)$. Liu and Meng [13] gave the representation formulas for spacelike curves on lightlike cones Λ^2 and Λ^3 . Babaarslan and Yaylı [2] examined the spacelike loxodromes on rotational surfaces which have spacelike meridians or timelike meridians. It is well-known that the Lorentzian sphere $S_1^2(r)$ and lightlike cone Λ^2 can be parameterized as ruled surfaces. Therefore, the research areas of ruled surfaces in the space $\mathbb{R}_1^3$ attract many researchers. For Example, Alías et al. [1] showed that B-scrolls over null curves are the ruled surfaces in 3-dimensional Lorentz-Minkowski space L^3 with null rulings whose Gauss map G satisfies the condition $\Delta G = \Lambda G$, where Δ denotes the Laplace operator of the surface and Λ is an endomorphism of L^3 . Kılıç et al. [9] examined the special null curves on the ruled surfaces and Liu [12] gave some characterizations of ruled surfaces with lightlike ruling in Minkowski 3-space. Choi and Kim [5] gave characterizations of some special helices and self associated curves

of a null curve and explicit representations of these curves, characterized general and slant helices in terms of their associated curves.

There are many models of hyperbolic geometry from non-Euclidean geometries. In these models, the first four postulates of Euclid are provided while the fifth postulate is not provided. The hyperboloid model, also known as the Minkowski model or the Lorentz model [15, 17, 19] is a model of 3-dimensional hyperbolic geometry in which points are represented by the points on the positive sheet $H^+(r)$ of Euclidean two-sheeted hyperboloid $H_0^2(r)$, the set of the unit timelike vectors of constant length r in 3-dimensional Minkowski space $\mathbb{R}_1^3$. The restriction of the quadric form induced by Lorentz metric $\langle ., . \rangle$ to the tangent plane of $H^+(r)$ is positive definite. So, it gives a Riemannian metric on $H^+(r)$ producing a model of 3-dimensional hyperbolic space H^3 .

The meridians of $H^+(r)$ are hyperbolic circles which are intersecting with timelike planes passing through the origin. The parallels are Euclidean circles which are intersecting with planes $z = k$, ($k > 1$). While the meridians are geodesic, the parallels are not geodesic. Since $H^+(r)$ is a spacelike surface, there must be spacelike spirals on the surface that cuts all meridians under a *constant spacelike angle*. The set of spacelike vectors with length r is called Lorentzian sphere and is denoted by $S_1^2(r)$. Note that Lorentzian sphere $S_1^2(r)$ is de Sitter Space-Time (See, [19]). The meridians of $S_1^2(r)$ are Lorentzian circles which are intersecting with planes including the origin and cover the z -axis; the parallels are Euclidean circles which are intersecting with planes $z = k$, ($k \in \mathbb{R}$). The parallels are not geodesic, except for the equator while the meridians are geodesics. Since $S_1^2(r)$ is a timelike surface, there must be three types of spirals on the surface: These are timelike spiral that cuts all meridians under a *constant hyperbolic angle*, spacelike spiral that cuts all meridians under a *constant timelike angle* and lightlike spiral that cuts all meridians under a *constant lightlike angle*. The meridians of Λ^2 are lightlike lines which are intersecting with lightlike planes passing through the origin. The parallels are Euclidean circles which are intersecting with planes $z = k$, ($k \in \mathbb{R}$). While the meridians are geodesic, the parallels are not geodesic. Since Λ^2 is a lightlike surface, there exists only spacelike spiral on the surface that cuts all parallels under a *constant Lorentzian angle* (angle with a measure of zero number).

In this paper, we obtain the differential equations of spirals on the surfaces of hyperboloid model $H^+(r)$, of the Lorentzian sphere $S_1^2(r)$ and of the lightlike cone Λ^2 in the Minkowski 3-space $\mathbb{R}_1^3$, and define the general parametrizations of causal spirals which are the solutions of the differential equations.

2 Basic Concepts

Let $\mathbb{R}_1^3$ be a Minkowski 3-space provided with natural Lorentzian metric given by

$$\langle ., . \rangle = dx_1^2 + dx_2^2 - dx_3^2, \quad (2.1)$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{R}_1^3$. An arbitrary vector $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$ is said to be spacelike (resp. timelike, lightlike) if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$ (resp. $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq 0$). The norm of a vector $\mathbf{v} \in \mathbb{R}_1^3$ is defined by $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$. Thus, a spacelike (timelike) vector $\mathbf{v}$ is unit if $\langle \mathbf{v}, \mathbf{v} \rangle = 1$ ($\langle \mathbf{v}, \mathbf{v} \rangle = -1$).

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the space $\mathbb{R}_1^3$. The Lorentzian cross product of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = (u_3v_2 - u_2v_3, u_1v_3 - u_3v_1, u_1v_2 - u_2v_1), \quad (2.2)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. For standard base vectors, we have

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = -\mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = -\mathbf{e}_2.$$

Let V be a 2-dimensional linear subspace $\mathbb{R}_1^3$. Then, there are three mutually exclusive possibilities for V :

- (i) V is said to be spacelike if the restriction $\langle ., . \rangle|_V$ of the Lorentzian metric on V is positive definite.
- (ii) V is said to be timelike if the restriction $\langle ., . \rangle|_V$ of the Lorentzian metric on V is Lorentzian, i.e, non-degenerate and of the signature (1). Then, V is a timelike plane.

(iii) V is lightlike (or null) if $\langle ., . \rangle|_V$ is degenerate.

The spheres of the space $\mathbb{R}_1^3$ are defined as follows:

The set of all spacelike vectors of radius $r > 0$ with origin-centered is called Lorentzian sphere of radius r and given by

$$S_1^2(r) = \{ \mathbf{v} \in \mathbb{R}_1^3 \mid \langle \mathbf{v}, \mathbf{v} \rangle = r^2 \}.$$

The sphere $S_1^2(r)$ is a Lorentzian 2-manifold of constant sectional curvature $1/r^2$. On the other hand, for $r > 0$, the quadric

$$H_0^2(r) = \{ \mathbf{v} \in \mathbb{R}_1^3 \mid \langle \mathbf{v}, \mathbf{v} \rangle = -r^2 \}$$

is called hyperbolic sphere of radius r . The sphere $H_0^2(r)$ is also a Riemannian 2-manifold of constant sectional curvature $-1/r^2$. This quadric has two connected components given by

$$\begin{aligned} H^+(r) &= \{ \mathbf{v} \in H_0^2(r) \mid \langle \mathbf{v}, \mathbf{e}_3 \rangle < 0 \}, \\ H^-(r) &= \{ \mathbf{v} \in H_0^2(r) \mid \langle \mathbf{v}, \mathbf{e}_3 \rangle > 0 \}. \end{aligned}$$

Each component is a simply connected Riemannian manifold of constant curvature $-1/r^2$. The set Λ^2 of all lightlike vectors

$$\Lambda^2 = \{ \mathbf{v} \in \mathbb{R}_1^3 - \{0\} \mid \langle \mathbf{v}, \mathbf{v} \rangle = 0 \}$$

is called lightlike cone of $\mathbb{R}_1^3$. This quadric has also two components given by

$$\begin{aligned} \Lambda^+ &= \{ \mathbf{v} \in \Lambda^2 \mid \langle \mathbf{v}, \mathbf{e}_3 \rangle < 0 \}, \\ \Lambda^- &= \{ \mathbf{v} \in \Lambda^2 \mid \langle \mathbf{v}, \mathbf{e}_3 \rangle > 0 \}. \end{aligned}$$

The components Λ^+ and Λ^- is called future cone and past cone, respectively. Then a ray in Λ^+ starting at the origin corresponds to a point on boundary of H^3 . The set of such rays form the sphere at infinity $S_\infty^2 = \partial H^3$. For an arbitrary point u in Λ^+ Heard [7] defines the horosphere as

$$h_u := \{ \mathbf{x} \in H^+ \mid \langle \mathbf{x}, \mathbf{u} \rangle = -1 \}$$

which inherits an Euclidean structure. Note that $\mathbf{u}$ is the position vector field of the point u .

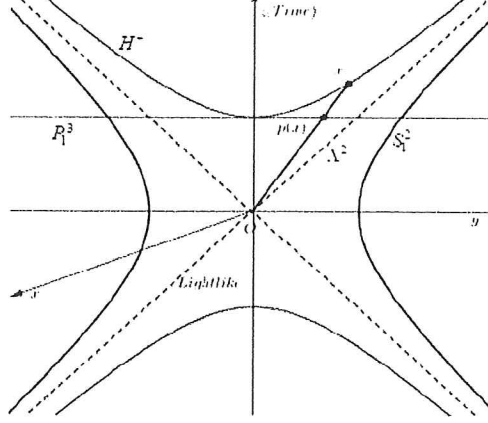


Figure 1: The space $\mathbb{R}^3_1$ equipped with the Lorentzian inner product $\langle \cdot, \cdot \rangle$ is 3 dimensional Lorentzian space.

Let denote the radial projection by p . Then the projection is given by

$$p : \{ \mathbf{x} \in \mathbb{R}^3_1 \mid x_3 \neq 0 \} \rightarrow P^3_1 = \{ \mathbf{x} \in \mathbb{R}^3_1 \mid x_3 = 1 \},$$

where P^3_1 is affine plane along the rays through the origin. The projection p is a homeomorphism from H^+ onto the 3-dimensional open unit ball B^3 in P^3_1 centered at the origin $(0, 0, 1)$ of P^3_1 , which give the projective model of H^3 . The affine plane P^3_1 contains B^3 and its set theoretic boundary ∂B^3 in P^3_1 , which is identified with S^2_∞ . Then we have $\bar{B}^3 = B^3 \cup \partial B^3$.

Now let define the geodesic plane $\mathbf{u}^\perp$ for an arbitrary point u in S^2_+ as

$$\mathbf{u}^\perp := \{ \mathbf{x} \in H^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle = 0 \}.$$

A point u also defines a half-space in H^3 given by

$$\Pi_u := \{ \mathbf{x} \in H^3 \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq 0 \},$$

where $\mathbf{u}$ is the position vector field of u .

Definition 2.1 The signed distance d between horosphere and a plane (respectively, point, horosphere) is the distance by which the horosphere extends past the plane (respectively, point, horosphere). The distance d may be positive, negative, or zero, as shown in (Figure 2). (For hyperbolic distances, see [7]).

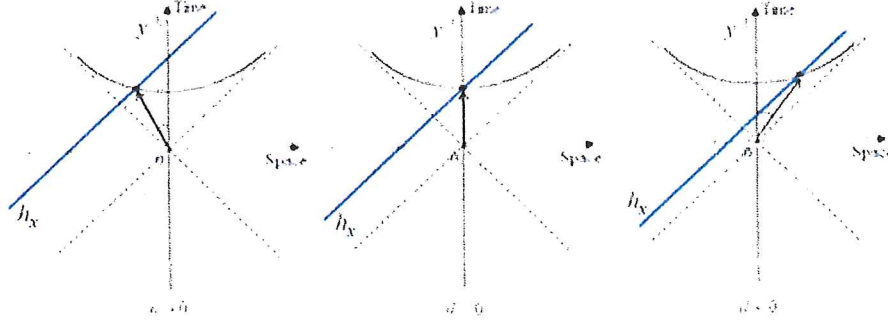


Figure 2: The signed distance from a plane to a horosphere is the distance d by which the horosphere extends to the plane.

3 The Angles in the Minkowski Space $\mathbb{R}_1^3$

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in the space $\mathbb{R}_1^3$. The angle between $\mathbf{u}$ and $\mathbf{v}$ is defined with respect to these causal characters as follows [3, 7, 14, 15, 17, 19]:

- (i) Let $\mathbf{u}$ and $\mathbf{v}$ be two timelike vectors. Then $|\langle \mathbf{u}, \mathbf{v} \rangle| \geq \|\mathbf{u}\| \|\mathbf{v}\|$ and equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are proportional. In this case, both vectors lie in the same cone and there is a unique real number $\theta \geq 0$ such that

$$\langle \mathbf{u}, \mathbf{v} \rangle = -\|\mathbf{u}\| \|\mathbf{v}\| \cosh \theta. \quad (3.1)$$

This number is called *hyperbolic angle* between the vectors $\mathbf{u}$ and $\mathbf{v}$.

- (ii) Let $\mathbf{u}$ and $\mathbf{v}$ be two spacelike vectors and they span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\| \cosh \theta. \quad (3.2)$$

This number is called *central angle* between the vectors $\mathbf{u}$ and $\mathbf{v}$.

- (iii) Let $\mathbf{u}$ and $\mathbf{v}$ be two spacelike vectors and they span a spacelike vector subspace. Then $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ and equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are proportional. In this case, there is a unique real number $\theta \geq 0$ such that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta. \quad (3.3)$$

This number is called *spacelike angle* between the vectors $\mathbf{u}$ and $\mathbf{v}$.

- (iv) Let $\mathbf{u}$ be a spacelike vector and $\mathbf{v}$ be a timelike vector. Then there is a unique real number $\theta \geq 0$ such that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \|\mathbf{v}\| \sinh \theta. \quad (3.4)$$

This number is called *timelike angle* between the vectors $\mathbf{u}$ and $\mathbf{v}$.

- (v) Two geodesic planes $\mathbf{u}^\perp$ and $\mathbf{v}^\perp$ do not intersect in H^2 but intersect in ∂H^2 if and only if

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = \cos 0 = 1. \quad (3.5)$$

In this case the *Lorentzian angle* between them is zero number. It must be note that the Lorentzian angles correspond to hyperbolic distances between two points on the hyperboloid model $H^+(r)$.

- (vi) Let $\mathbf{u} \in \Lambda^+$ and $\mathbf{v} \in S_1^2$. The *lightlike angle* between them is signed distance d between $\mathbf{v}$ and $h_{\mathbf{u}}$. This angle satisfies the equality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = e^{-d}.$$

- (vii) Let $\mathbf{u} \in \Lambda^+$ and $\mathbf{v} \in H^+$. The *lightlike angle* between them is signed distance d between $\mathbf{v}^\perp$ and $h_{\mathbf{u}}$. This angle satisfies the equality

$$\langle \mathbf{u}, \mathbf{v} \rangle = -e^{-d}.$$

- (viii) Let $\mathbf{u}, \mathbf{v} \in \Lambda^+$. The *lightlike angle* between them is signed distance d between $h_{\mathbf{u}}$ and $h_{\mathbf{v}}$. This angle satisfies the equality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = 2e^{-d}.$$

(For the geometrical interpretations of angles in Lorentzian 3-space, see [7, 17]).

4 The Spirals on the Hyperboloid Model $H^+(r)$

The hyperbolic sphere with center O and radius r is the surface given by

$$H_0^2(r) = \left\{ (x, y, z) \in \mathbb{R}_1^3 \mid x^2 + y^2 - z^2 = -r^2 \right\}.$$

We observe that the set $H_0^2(r)$ has exactly two connected components. Thus, the hyperboloid model of hyperbolic geometry from non-Euclidean geometries in the Lorentz-Minkowski space is denoted by

$$H^+(r) = \left\{ (x, y, z) \in \mathbb{R}_1^3 \mid x^2 + y^2 - z^2 = -r^2, z > 0 \right\}.$$

This is a spacelike surface and called a hyperbolic plane [7, 17, 19]. We shall compute the first fundamental form of $H^+(r)$ at a point of the coordinate neighborhood given by parametrization

$$x(u, v) = (r \sinh u \cos v, r \sinh u \sin v, r \cosh u) \quad (4.1)$$

where $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$. First, observe that

$$\begin{cases} \mathbf{x}_u(u, v) = (r \cosh u \cos v, r \cosh u \sin v, r \sinh u) \\ \mathbf{x}_v(u, v) = (-r \sinh u \sin v, r \sinh u \cos v, 0). \end{cases} \quad (4.2)$$

Hence,

$$E(u, v) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = r^2, F(u, v) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0, G(u, v) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = r^2 \sinh^2 u. \quad (4.3)$$

If $\mathbf{w}$ is a tangent vector to the sphere at the point $x(u, v)$, then $\mathbf{w}$ is given in the basis associated to $x(u, v)$ by

$$\mathbf{w} = a\mathbf{x}_u + b\mathbf{x}_v.$$

Thus, the square of the length of $\mathbf{w}$ is given by

$$|\mathbf{w}|^2 = I(\mathbf{w}) = Ea^2 + 2Fab + Gb^2 = a^2r^2 + b^2r^2\sinh^2 u.$$

Let us determine the spirals in this coordinate neighborhood of the hyperboloid model $H^+(r)$ which make a constant angle θ with the meridians ($v = \text{const.}$). In this case, we give definitions as follows:

Definition 4.1 Let $\alpha_\lambda(t)$ be a differentiable curve on the surface of the hyperboloid model $H^+(r)$. The curve $\alpha_\lambda(t)$ is called *hyperbolic spherical spiral* if $\alpha_\lambda(t)$ cuts all meridians of $H^+(r)$ with a constant spacelike angle.

Theorem 4.2 Let θ be constant spacelike angle between the meridians and the spacelike curve $\alpha_\lambda(t)$ and assume that $\lambda = \cot \theta$. Then the parametrization of all spirals on the surface of the hyperboloid model $H^+(r)$ is given by

$$\alpha_\lambda(t) = \left(\frac{2re^{\lambda t}}{e^{2\lambda t} - 1} \cos t, \frac{2re^{\lambda t}}{e^{2\lambda t} - 1} \sin t, r \frac{e^{2\lambda t} + 1}{e^{2\lambda t} - 1} \right). \quad (4.4)$$

Proof. We may assume that the required curve

$$\alpha_\lambda(t) = x(u(t), v(t)) = (r \sinh u(t) \cos v(t), r \sinh u(t) \sin v(t), r \cosh u(t))$$

is the image by x of a curve $(u(t), v(t))$ of the uv -plane. In this case we have

$$\alpha'_\lambda(t) = (ru'(t) \cosh u(t) \cos v(t) - rv'(t) \sinh u(t) \sin v(t), \\ ru'(t) \cosh u(t) \sin v(t) + rv'(t) \sinh u(t) \cos v(t), ru'(t) \sinh u(t)).$$

At the point $x(u, v)$ where the curve meets the meridians ($v = \text{const.}$), we have

$$\cos \theta = \frac{|\langle \mathbf{x}_u, \alpha'_\lambda(t) \rangle|}{\|\mathbf{x}_u\| \|\alpha'_\lambda(t)\|} = \frac{u'}{\sqrt{(u')^2 + (v')^2 \sinh^2 u}}, \quad (4.5)$$

since in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ the vector $\alpha'_\lambda(t)$ has coordinates (u', v') and the vector $\mathbf{x}_u$ has coordinates $(1, 0)$. By (12) we obtain the differential equation as

$$(u')^2 \tan^2 \theta - (v')^2 \sinh^2 u = 0$$

or

$$\frac{du}{\sinh u} = \pm \frac{dv}{\tan \theta}. \quad (4.6)$$

Hence, by integration we obtain the differential equation of the hyperbolic spherical spirals as

$$\ln \left(\tanh \frac{u}{2} \right) = \pm (v + c) \cot \theta, \quad (4.7)$$

where the constant of integration c is to be determined by giving one point $x(u_0, v_0)$ which the curve passes. By taking the signature $+$, $c = 0$ and $v(t) = t$, the solution of the differential equation (14) gives

$$u(t) = 2 \tanh^{-1} (e^{(\cot \theta)t}) . \quad (4.8)$$

By the equality (15) it is seen that

$$\sinh u(t) = \frac{2e^{(\cot \theta)t}}{e^{2(\cot \theta)t} - 1}, \quad \cosh u(t) = \frac{e^{2(\cot \theta)t} + 1}{e^{2(\cot \theta)t} - 1}. \quad (4.9)$$

By writing the equalities (16) in (8) we obtain the parametrization (11) and this completes the proof (See, Figure 3). ■

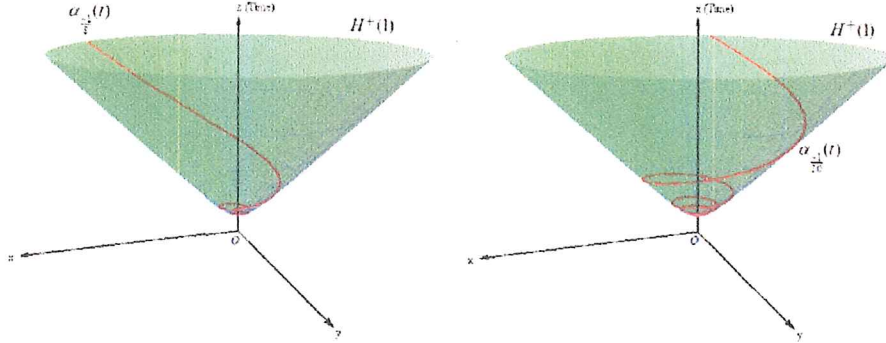


Figure 3: Two spirals on the surface of the hyperboloid model $H^+(1)$.

5 The Spirals on the Lorentzian Sphere $S_1^2(r)$

Since the Lorentzian sphere is a timelike surface, there are three types of geodesic called spacelike, timelike and lightlike (null) geodesics. Similarly, there must be three kinds of spirals that cut the meridians under fixed hyperbolic, timelike and Lorentzian angles. In this section, we find the differential equations of causal spirals that provide certain conditions and we give the parametrizations of the timelike, spacelike and lightlike spirals

which are the solutions of these equations.

We know that a parametrization of the Lorentzian sphere $S_1^2(r)$ is given by

$$y(u, v) = (r \cosh u \cos v, r \cosh u \sin v, r \sinh u) \quad (5.1)$$

where $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$. First, observe that

$$\begin{cases} \mathbf{y}_u(u, v) = (r \sinh u \cos v, r \sinh u \sin v, r \cosh u) \\ \mathbf{y}_v(u, v) = (-r \cosh u \sin v, r \cosh u \cos v, 0). \end{cases} \quad (5.2)$$

Hence,

$$E(u, v) = \langle \mathbf{y}_u, \mathbf{y}_u \rangle = -r^2, F(u, v) = \langle \mathbf{y}_u, \mathbf{y}_v \rangle = 0, G(u, v) = \langle \mathbf{y}_v, \mathbf{y}_v \rangle = r^2 \cosh^2 u. \quad (5.3)$$

If $\mathbf{w}$ is a tangent vector to the sphere at the point $y(u, v)$, then $\mathbf{w}$ is given in the basis associated to $y(u, v)$ by

$$\mathbf{w} = a\mathbf{y}_u + b\mathbf{y}_v.$$

Thus, the square of the length of $\mathbf{w}$ is given by

$$|\mathbf{w}|^2 = I(\mathbf{w}) = Ea^2 + 2Fab + Gb^2 = -a^2r^2 + b^2r^2 \cosh^2 u.$$

Let us determine the causal spirals in this coordinate neighborhood of the Lorentzian sphere $S_1^2(r)$ which make a constant angle θ with the meridians ($v = \text{const.}$). In this case, we give definitions as follows:

Definition 5.1 Let $\beta_\lambda(t)$ be a differentiable curve on the surface of the Lorentzian sphere $S_1^2(r)$. The curve $\beta_\lambda(t)$ is called Lorentzian timelike (resp., spacelike, lightlike) spherical spiral if $\beta_\lambda(t)$ cuts all meridians of $S_1^2(r)$ with a constant hyperbolic (resp., timelike, Lorentzian) angle.

Theorem 5.2 The parametrization of all timelike (resp., spacelike, lightlike) spirals on the surface of the Lorentzian sphere $S_1^2(r)$ is given by

$$\beta_\lambda(t) = (r \sec(\lambda t) \cos t, r \sec(\lambda t) \sin t, -r \tan(\lambda t)), \quad (5.4)$$

where $\lambda = \coth \theta$ (resp., $= \tanh \theta$, $= \pm 1$) and θ is constant hyperbolic (resp., timelike, Lorentzian) angle between meridians and the curve $\beta_\lambda(t)$.

Proof. We may assume that the required causal curve

$$\beta_\lambda(t) = y(u(t), v(t)) = (r \cosh u(t) \cos v(t), r \cosh u(t) \sin v(t), r \sinh u(t))$$

is the image by y of a curve $(u(t), v(t))$ of the uv plane. In this case, we have

$$\begin{aligned} \beta'_\lambda(t) = & (r u'(t) \sinh u(t) \cos v(t) - r v'(t) \cosh u(t) \sin v(t), \\ & r u'(t) \sinh u(t) \sin v(t) + r v'(t) \cosh u(t) \cos v(t), r u'(t) \cosh u(t)). \end{aligned}$$

(i) Let the required causal curve $\beta_\lambda(t)$ be timelike curve.

At the point $y(u, v)$ where the timelike curve meets the meridians ($v = \text{const.}$), we have

$$-\cosh \theta = \frac{\langle \mathbf{y}_u, \beta'_\lambda(t) \rangle}{\|\mathbf{y}_u\| \|\beta'_\lambda(t)\|} = \frac{-u'}{\sqrt{(u')^2 - (v')^2 \cosh^2 u}}, \quad (5.5)$$

since in the basis $\{\mathbf{y}_u, \mathbf{y}_v\}$ the vector $\beta'_\lambda(t)$ has coordinates (u', v') and the vector $\mathbf{y}_u$ has coordinates $(1, 0)$. By (21) we obtain the differential equation as

$$(u')^2 \tanh^2 \theta - (v')^2 \cosh^2 u = 0$$

or

$$\frac{du}{\cosh u} = \pm \frac{dv}{\tanh \theta}. \quad (5.6)$$

Hence, by integration we obtain the differential equation of the Lorentzian spherical timelike spirals as

$$2 \tan^{-1} \left(\tanh \frac{u}{2} \right) = \pm (v + c) \coth \theta, \quad (5.7)$$

where the constant of integration c is to be determined by giving one point $y(u_0, v_0)$ which the curve passes. By taking the signature $+$, $c = 0$ and $v(t) = t$, the solution of the differential equation (23) gives

$$u(t) = 2 \tanh^{-1} \left(\tan \left(\frac{\coth \theta}{2} t \right) \right). \quad (5.8)$$

By the equality (24) it is seen that

$$\sinh u(t) = -\tan(\lambda t), \quad \cosh u(t) = \sec(\lambda t). \quad (5.9)$$

By writing the equalities (25) in (17) we obtain the parametrization (20) and this completes the proof of (i) (See, Figure 4).

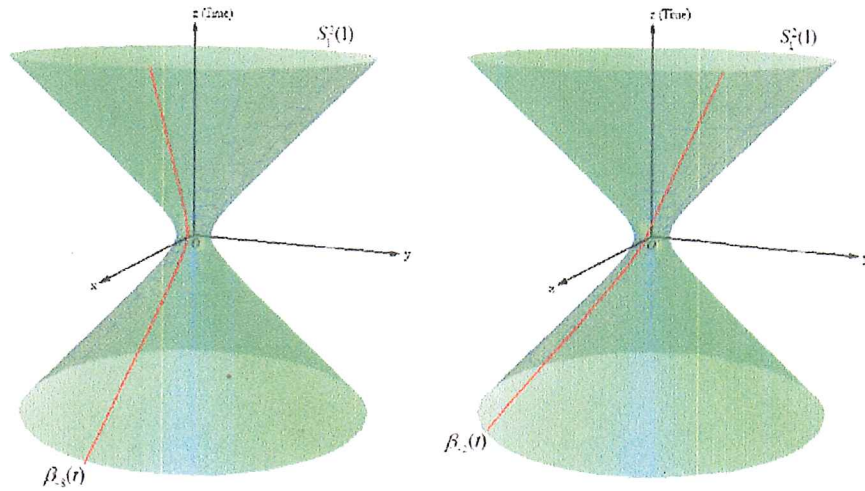


Figure 4: Two Timelike spirals on the surface of the Lorentzian sphere $S_1^2(1)$.

- (ii) Let the required causal curve $\beta_\lambda(t)$ be spacelike curve.

At the point $y(u, v)$ where the timelike curve meets the meridians ($v = \text{const.}$), we have

$$\sinh \theta = \frac{|\langle \mathbf{y}_u, \beta'_\lambda(t) \rangle|}{\|\mathbf{y}_u\| \|\beta'_\lambda(t)\|} = \frac{u'}{\sqrt{-(u')^2 + (v')^2 \cosh^2 u}}, \quad (5.10)$$

since in the basis $\{\mathbf{y}_u, \mathbf{y}_v\}$ the vector $\beta'_\lambda(t)$ has coordinates (u', v') and the vector $\mathbf{y}_u$ has coordinates $(1, 0)$. By (26) we obtain the differential equation as

$$(u')^2 \coth^2 \theta - (v')^2 \cosh^2 u = 0$$

or

$$\frac{du}{\cosh u} = \pm \frac{dv}{\coth \theta}. \quad (5.11)$$

Hence, by integration we obtain the differential equation of the Lorentzian spherical spacelike spirals as

$$2 \tan^{-1} \left(\tanh \frac{u}{2} \right) = \pm (v + c) \tanh \theta, \quad (5.12)$$

where the constant of integration c is to be determined by giving one point $y(u_0, v_0)$ which the curve passes. By taking the signature $+$, $c = 0$ and $v(t) = t$, the solution of the equation (28) gives

$$u(t) = 2 \tanh^{-1} \left(\tan \left(\frac{\tanh \theta}{2} t \right) \right). \quad (5.13)$$

By the equality (29) it is seen that

$$\sinh u(t) = -\tan(\lambda t), \quad \cosh u(t) = \sec(\lambda t). \quad (5.14)$$

By writing the equalities (30) in (17) we obtain the parametrization (20) and this completes the proof of (ii) (See, Figure 5).

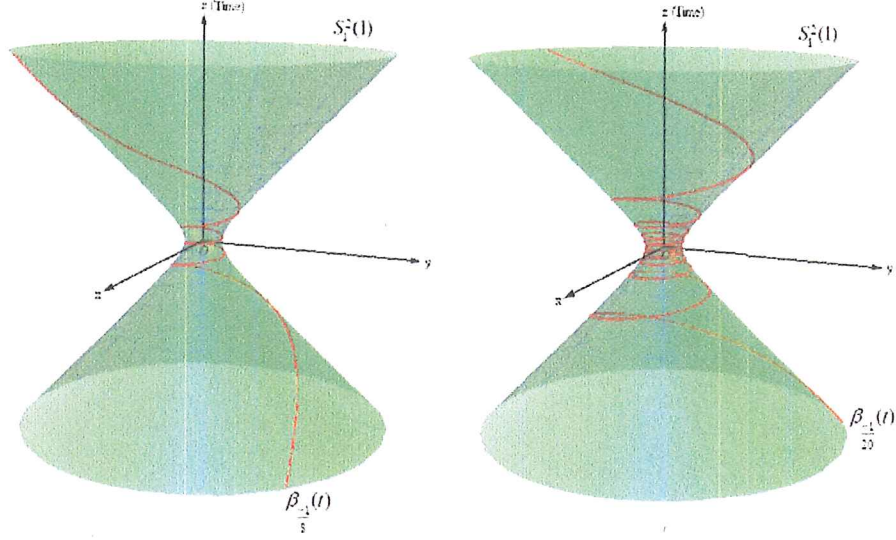


Figure 5: Two Spacelike spirals on the surface of the Lorentzian sphere $S_1^2(1)$.

- (iii) Let the required causal curve $\beta_\lambda(t)$ be lightlike (null) curve. In this case, we have $\lambda = \pm 1$. Thus the parametrizations of lightlike spirals are given by

$$\beta_1(t) = (r, r \tan t, -r \tan t) \text{ and } \beta_{-1}(t) = (r, r \tan t, r \tan t). \quad (5.15)$$

The parametrizations in (31) show two lightlike lines passing from the point $P(1, 0, 0)$. This means that all lightlike spirals on the Lorentzian sphere $S_1^2(r)$ are lightlike lines (geodesics) (See, Figure 6).

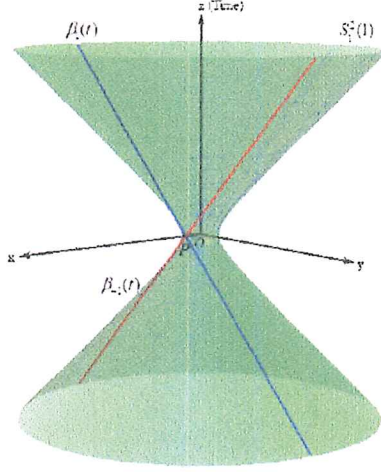


Figure 6: Two Lightlike spirals on the surface of the Lorentzian sphere $S_1^2(1)$.

Consequently, (i), (ii) and (iii) complete the proof. ■

6 The Spirals on the Lightlike Cone Λ^2

We know that a parametrization of the lightlike cone Λ^2 is given by

$$z(u, v) = (v \cos u, v \sin u, v) \quad (6.1)$$

where $0 \leq u \leq 2\pi$ and $v \in \mathbb{R}$. First, observe that

$$\begin{cases} z_u(u, v) = (-v \sin u, v \cos u, 0) \\ z_v(u, v) = (\cos u, \sin u, 1). \end{cases} \quad (6.2)$$

Hence,

$$E(u, v) = \langle z_u, z_u \rangle = v^2, \quad F(u, v) = \langle z_u, z_v \rangle = 0, \quad G(u, v) = \langle z_v, z_v \rangle = 0. \quad (6.3)$$

If w is a tangent vector to the lightlike cone at the point $z(u, v)$, then w is given in the basis associated to $z(u, v)$ by

$$w = az_u + bz_v.$$

Thus, the square of the length of $\mathbf{w}$ is given by

$$|\mathbf{w}|^2 = I(\mathbf{w}) = Ea^2 + 2Fab + Gb^2 = v^2a^2.$$

Let us determine the curves in this coordinate neighborhood of the lightlike cone Λ^2 which make a constant angle with the meridians (or parallels). In this case, we give definitions as follows:

Definition 6.1 *Let $\gamma_\lambda(t)$ be a differentiable curve on the surface of the lightlike cone Λ^2 . The curve $\gamma_\lambda(t)$ is called lightlike conical spiral if $\gamma_\lambda(t)$ cuts all parallels (or lightlike lines) of Λ^2 with zero angle.*

Now, we give parametrization of the lightlike conical spirals with constant speed as follows:

Theorem 6.2 *The parametrization of all constant speed spirals on the surface of the lightlike cone Λ^2 is given by*

$$\gamma_\lambda(t) = (t \cos(\lambda \ln t), t \sin(\lambda \ln t), t). \quad (6.4)$$

where $\lambda \in \mathbb{R}^+$ denotes the constant speed of $\gamma_\lambda(t)$.

Proof. We may assume that the required curve

$$\gamma_\lambda(t) = z(u(t), v(t)) = (v(t) \cos u(t), v(t) \sin u(t), v(t))$$

is the image by z of a curve $(u(t), v(t))$ of the uv plane. In this case we have

$$\gamma'_\lambda(t) = (v'(t) \cos u(t) - u'(t)v(t) \sin u(t), \\ v'(t) \sin u(t) + u'(t)v(t) \cos u(t), v'(t)).$$

At the point $z(u, v)$ where the curve meets the parallels ($v = \text{const.}$), we have

$$\frac{|\langle \mathbf{z}_u, \gamma'_\lambda(t) \rangle|}{\|\mathbf{z}_u\| \|\gamma'_\lambda(t)\|} = 1, \quad (6.5)$$

since in the basis $\{\mathbf{z}_u, \mathbf{z}_v\}$ the vector $\gamma'_\lambda(t)$ has coordinates (u', v') and the vector $\mathbf{z}_u$ has coordinates $(1, 0)$. By (36) we satisfy (7), but (36)

does not give any differential equation. On the other hand, we see that $\|\gamma'_\lambda(t)\| = u'(t)v(t)$. In this case, there is a way to obtain a differential equation: the spiral $\gamma_\lambda(t)$ can be thought as a constant speed curve. Thus, the inner product $|\langle z_u, \gamma'_\lambda(t) \rangle|$ becomes constant (for details, see [7]). Then, the differential equation is given by

$$u'v = \lambda$$

or

$$du = \lambda \frac{dv}{v}. \quad (6.6)$$

Hence, by integration we obtain the differential equation of the lightlike conical spiral as

$$u = \lambda \ln(v) + c, \quad (6.7)$$

where the constant of integration c is to be determined by giving one point $z(u_0, v_0)$ which the curve passes. Specially, we assume that $c = 0$ and $v(t) = t$. Thus, we have $u(t) = \lambda \ln t$. By writing these equalities in (32) we obtain the parametrization (35) and this completes the proof (See, Figure 7). ■

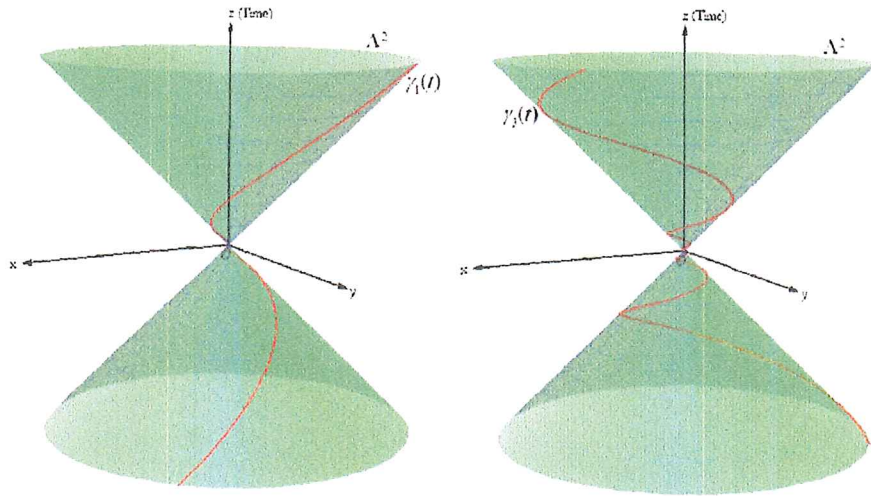


Figure 7: Two spirals on the surface of the lightlike cone Λ^2 .

7 Conclusion

In this paper, we obtained differential equations of all spirals on the positive hyperbolic sphere $H^+(r)$, the Lorentzian sphere $S_1^2(r)$ and the positive lightlike cone Λ^+ of the Lorentz-Minkowski space $\mathbb{R}_1^3$, and gave the general parametrizations of these curves. In the next study, we will define spiral curves on some special surfaces of the space $\mathbb{R}_1^3$ and examine their geometries.

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